MATH 121, Calculus I — Exam II (Fall 2013)

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Instructor: Bozenna Pasik-Duncan **Section:** ______

1. **Do not open this exam until you are told to do so.**
2. This exam has of 10 pages, including this cover sheet. Do not separate the pages of the exam; if they do become separated, write your name on every page and point it out to your instructor when you hand in the exam.
3. This exam has a total value of 200 points and consists of 14 problems that are divided between two parts; the first part contains 10 multiple-choice problems and the second part contains 4 "show your work" problems. Additionally, the last page of the exam contains an extra-credit problem that is worth 20 points.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about problems during the exam.
5. This is strictly a closed-book exam and the use of technology (including calculators, phones, tablets, and laptops) is prohibited. A single 3" × 5" note is allowed.
6. No extra paper is allowed and only the work shown on the front side of each provided page of the exam will be graded.
7. You must use the methods learned in this course to solve all problems.

Problem	1	2	3	4	5	6	7	8	9	10
Points	10	10	10	10	10	10	10	10	10	10
Score										

Problem	11	12	13	14	Extra Credit
Points	30	30	20	20	20
Score					

Exam II total score

Honor code: "I have not cheated on this exam and I am not aware that anyone else has cheated on this exam."

Signatures: ______________________ (at time of submission).

______________________ (upon receipt of graded exam).

Multiple-Choice Questions

Instructions: Place the appropriate letter for your answer for each problem in the blank box that is provided. Correct answers do not require work to receive full credit. However, partial credit can be awarded to incorrect answers based on work shown in the adjacent blank space. Hence, you are strongly advised to show work for each problem.

1. **[10 points]** Find the absolute extreme values of the function $f(x) = x^3 - 12x + 5$ on the closed interval $[-1, 3]$.

   (A) -11 and 21

   (B) -4 and 16

   (C) -11 and 16

   (D) -11 and -4

   Answer: C

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$

$$f'(x) = 0 \iff x^2 - 4 = 0$$

$$x_1 = -2, \quad x = 2$$

$$f(-1) = -1 + 12 + 5 = 16$$

$$f(3) = 27 - 36 + 5 = -4$$

$$\cancel{f(-2) = -8 + 24 + 5 = 21}$$

$$f(2) = 8 - 24 + 5 = -11$$

not in domain therefore not critical point

2. **[10 points]** Find the point(s) of inflection of the function $f(x) = x^4 - 2x^3$.

   (A) $(0, 0)$

   (B) $(1, -1)$

   (C) The function has no points of inflection.

   (D) $(0, 0)$ and $(1, -1)$

   Answer: D

$$f'(x) = 4x^3 - 6x^2$$

$$f''(x) = 12x^2 - 12x = 12x(x - 1)$$

$$f''(x) = 0 \text{ iff } x = 0, \; x = 1$$

f'': $+ \quad 0 \quad - \quad 1 \quad +$

3. **[10 points]** Find the critical number(s) of the function $f(x) = x^{2/3} - \frac{1}{5}x^{5/3}$.

(A) The function has no critical number.

(B) $x = 0$

(C) $x = 0$ and $x = 2$

(D) $x = \sqrt[3]{2}$

Answer: C

$$f'(x) = \frac{2}{3}x^{-1/3} - \frac{1}{5}\cdot\frac{5}{3}x^{2/3}$$

$$f'(x) = 0 \iff \frac{2}{3}\frac{1}{\sqrt[3]{x}} - \frac{1}{3}\left(\sqrt[3]{x^2}\right) = 0$$

$$2 - x = 0$$

$\boxed{x = 2}$ and $\boxed{x = 0}$

f' is undefined

4. **[10 points]** Let $C(x) = 800 + 0.04x + 0.0002x^2$ be the cost in dollars of producing x units of a certain product. Find the production level that minimizes the average cost per unit.

(A) $x = 200$ units

(B) $x = 400$ units

(C) $x = 4000$ units

(D) $x = 2000$ units

Answer: D

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{800}{x} + 0.04 + 0.0002x$$

$D_{\bar{C}} = \{x : x > 0\}$

$$\bar{C}'(x) = -\frac{800}{x^2} + 0.0002$$

$$\bar{C}'(x) = 0 \text{ iff } -\frac{800}{x^2} + 0.0002 = 0$$

$$-800 + 0.0002x^2 = 0$$

$$\iff x^2 = \frac{800}{0.0002}$$

$$x^2 = \frac{400}{0.0001}$$

$$x^2 = 4000000$$

$\boxed{x = 2000}$ unit

5. [**10 points**] Find $\int_{-1}^{4} f(x)dx$, provided that $\int_{-1}^{4}(2f(x) - 7)dx = -31$.

(A) 31

(B) -12

(C) 2

(D) 4

Answer: C

$$\int_{-1}^{4}(2f(x) - 7)\,dx = -31$$

$$2\int_{-1}^{4} f(x)\,dx - 7\int_{-1}^{4} dx = -31$$

$$2\int_{-1}^{4} f(x)\,dx - 7\left[x\right]_{-1}^{4} = -31$$

$$2\int_{-1}^{4} f(x)\,dx - 7\left[4+1\right] = -31$$

$$2\int_{-1}^{4} f(x)\,dx - 35 = -31 \Rightarrow 2\int_{-1}^{4} f(x)\,dx = 4$$

$$\Rightarrow \boxed{\int_{-1}^{4} f(x)\,dx = 2}$$

6. [**10 points**] Find the asymptote(s) of the function $f(x) = \dfrac{x^2 - 25}{x^2}$.

(A) $x = 0$

(B) $y = 1$

(C) The function has no asymptotes.

(D) $x = 0$ and $y = 1$

Answer: D

- $$\lim_{x \to 0} \frac{x^2 - 25}{x^2} = -\infty$$

$\Rightarrow$ $x = 0$ is a vertical asymptote

- $$\lim_{x \to \pm\infty} \frac{x^2 - 25}{x^2} = \lim_{x \to \pm\infty} \frac{\cancel{x^2}\left(1 - \frac{25}{x^2}\right)}{\cancel{x^2}} = 1$$

$\Rightarrow$ $y = 1$ is ~~an~~ horizontal asymptote

7. [**10 points**] Evaluate the integral $\int_0^3 f(x)dx$, where the function f is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 2; \\ 4, & \text{if } x \geq 2. \end{cases}$$

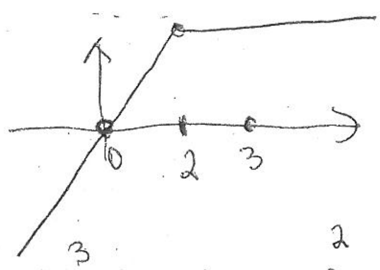
(A) 4

(B) 12

(C) 8

(D) 9

Answer: C



$$\int_0^3 f(x)dx = \int_0^2 2x\,dx + \int_2^3 4\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^2 + 4\left[x\right]_2^3$$

$$= 2(2-0) + 4(3-2) = 4+4 = 8$$

8. [**10 points**] Find $F'(x)$, provided that $F(x) = \int_1^x (t^2 - 2t + 3)dt$.

(A) $t^2 - 2t + 3$

(B) $x^2 - 2x + 3$

(C) $\dfrac{t^3}{3} - 2\dfrac{t^2}{2} + 3t$

(D) $x^2 - 2x + 1$

Answer: B

$$F'(x) = x^2 - 2x + 3$$

9. [**10 points**] Find $F'(x)$, provided that $F(x) = \int_0^{x^2} \cos t \, dt$.

(A) $\cos t$

(B) $\cos(x^2)$

(C) $\cos(x^2) - 1$

(D) $2x\cos(x^2)$

Answer: D

$$F'(x) = (\cos x^2) \cdot 2x$$
$$= 2x\cos(x^2)$$

10. [**10 points**] Find $\int x\cos(x^2)\,dx$.

(A) $\sin(x^2) + C$

(B) $\cos(x^2) + C$

(C) $x\sin(x^2)$

(D) $\frac{1}{2}\sin(x^2) + C$

Answer: D

$$F(x) = \frac{1}{2}\sin(x^2) + c$$
$$F'(x) = \frac{1}{2}\cos(x^2) \cdot 2x$$
$$= x\cos(x^2)$$

or let $u = x^2$

$du = 2x\,dx$

$$\Rightarrow \frac{1}{2}\int 2x\cos(x^2)\,dx = \frac{1}{2}\int \cos u\,du$$
$$= \frac{1}{2}\sin u = \frac{1}{2}\sin(x^2) + C$$

"Show Your Work" Problems

Instructions: Please show all necessary work and provide full justification for each answer. Place a box around each answer.

11. **[30 points]** Let $f(x) = 2x^3 - 3x^2 - 12x$, and observe that the first two derivatives of the function f are given by $f'(x) = 6(x-2)(x+1)$ and $f''(x) = 6(2x-1)$.

(a) State the domain f and find the limits of f as $x \to +\infty$ and $x \to -\infty$.

5 pts

$D_f = \mathbb{R} = (-\infty, +\infty)$

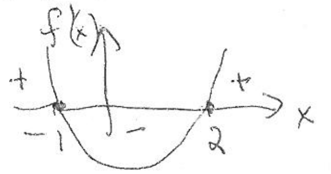
$\lim_{x\to-\infty} x^3\left(2 - \underbrace{\frac{3}{x}}_{\to 0} - \underbrace{\frac{12}{x^2}}_{\to 0}\right) = -\infty$

$\lim_{x\to+\infty} x^3\left(2 - \frac{3}{x} - \frac{12}{x^2}\right) = +\infty$

(b) On what interval(s) is f increasing? On what interval(s) is f decreasing? List any extrema of f.

10 pts

$f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1) = 0$ iff $x = -1$, $x = 2$

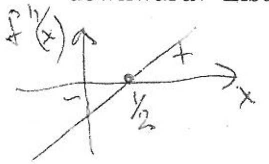


$f_{max}(-1) = 2\cdot(-1)^3 - 3(-1)^2 - 12(-1) = -2 - 3 + 12 =$

$f_{min}(2) = 2\cdot 8 - 3\cdot 4 - 12\cdot 2 = 16 - 12 - 24 = -20$

(c) On what interval(s) is f is concave upward? On what interval(s) is f concave downward? List any point(s) of inflection of f.

10 pts



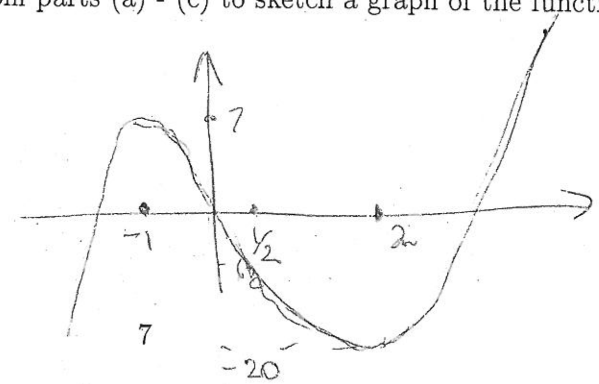
$f''(x) = 0$ iff $x = \frac{1}{2}$

$\left.\begin{array}{l} f''(x) > 0 \text{ for } x > \frac{1}{2} \\ f''(x) < 0 \text{ for } x < \frac{1}{2} \end{array}\right) \Rightarrow \left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right)$ is the point of inflection

$f\left(\frac{1}{2}\right) = 2\cdot\frac{1}{8} - 3\cdot\frac{1}{4} - 6 = \frac{1}{4} - \frac{3}{4} - \frac{24}{4} = -\frac{1}{2} - \frac{12}{2} = -\frac{13}{2} = -6\frac{1}{2}$

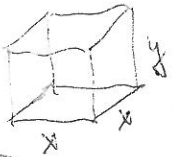
(d) Use the information from parts (a) - (c) to sketch a graph of the function f.

5 pts



12. **[30 points]** A manufacturer wants to design an open box with a square base and a fixed surface area of 108 in^2. Complete the following steps to determine the dimensions that will produce a box of maximum volume.

(a) Express the volume, V, of the box as a function of x, the side length of its base.



$V = x^2 \cdot y$

$S = x^2 + 4xy$

$108 = x^2 + 4xy \Rightarrow y = \frac{108 - x^2}{4x}$

10pts

$V(x) = x^2 \cdot \frac{108 - x^2}{4x} = \frac{1}{4}x(108 - x^2) = -\frac{1}{4}x^3 + \frac{108x}{4}$

$V(x) = -\frac{1}{4}x^3 + 27x$

(b) Determine the interval over which V is to be maximized (i.e. find the restrictions on the side length x).

5pts

$x \geq 0$ and $y \geq 0 \iff \frac{108 - x^2}{4x} \geq 0$

$\iff 108 - x^2 \geq 0$

$x^2 \leq 108$

$x \leq \sqrt{108}$

$\boxed{0 \leq x \leq \sqrt{108}}$

(c) Find the dimensions of the box that will maximize its volume. Justify your answer.

15pts

$V'(x) = -\frac{3}{4}x^2 + 27$

$V'(x) = 0$ iff $-\frac{3}{4}x^2 + 27 = 0$

$-3x^2 + 108 = 0$

$x^2 = \frac{108}{3} = 36 \iff x = \sqrt{\frac{108}{3}}$

$x = \sqrt{36}$

10pts

$y = \frac{108 - \cancel{\frac{108}{3}}\,36}{\cancel{36}\,24} = \frac{\cancel{324 - 108}}{\cancel{36}} = \frac{72}{\cancel{36}\,24} = \cancel{2}\,3\text{ in}$ $\boxed{x = 6\text{ in}}$

5pts justification

$V''(x) = -\frac{6}{4}x$

$V''(6) = -\frac{6}{4} \cdot 6 < 0 \Rightarrow$ max

~~$V(0) = 0$ and $V(\sqrt{108}) = 0$~~

$\lim V(x) = 0$ $\lim_{x \to \sqrt{108}} V(x)$

$\Rightarrow$ $\boxed{x = 6\text{ in and } y = \cancel{2}\,3\text{ in}}$ maximize the volume of the box.

13. [**20 points**] The acceleration of a particle is given by $a(t) = 6 - 6t$ for $t \geq 0$.

(a) Find the velocity function, $v(t)$, provided that $v(0) = 0$.

10 pts

$$v(t) = \int a(t)\,dt = \int (6 - 6t)\,dt = 6t - 3t^2 + C_1$$

$$0 = v(0) = 6\cdot 0 - 6\cdot 0^2 + C_1 \Rightarrow C_1 = 0$$

$$\Rightarrow v(t) = 6t - 3t^2$$

(b) Find the position function, $s(t)$, provided that $s(0) = 3$.

10 pts

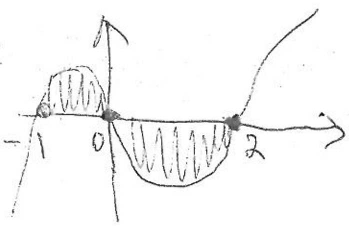
$$s(t) = \int v(t)\,dt = \int (6t - 3t^2)\,dt = 3t^2 - t^3 + C_2$$

$$3 = s(0) = 3\cdot 0^2 - 0^3 + C_2 \Rightarrow C_2 = 3$$

$$\Rightarrow \boxed{s(t) = 3t^2 - t^3 + 3}$$

14. [**20 points**] Consider the function $f(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$.

(a) Sketch a graph of the region bounded by the x-axis, the lines $x = -1$ and $x = 2$, and the curve $y = f(x)$. Express the area of this region as a sum of two integrals.



$$A = \int_{-1}^{0} f(x)\,dx + \int_{0}^{2} (-f(x))\,dx$$

10 pts

(b) Evaluate the integral expression from part (a) to find the area of the region.

10 pts

$$A = \int_{-1}^{0} (x^3 - x^2 - 2x)\,dx + \int_{0}^{2} (-x^3 + x^2 + 2x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} + \left[-\frac{x^4}{4} + \frac{x^3}{3} + x^2\right]_{0}^{2}$$

$$= \left(0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right)\right) + \left(-\cancel{4} + \frac{8}{3} + \cancel{4} - 0\right)$$

$$= -\frac{1}{4} - \frac{1}{3} + 1 + \frac{8}{3} = -\frac{1}{4} + 1 + \frac{7}{3} = \frac{3}{4} + \frac{7}{3} = \frac{9+28}{12} =$$

$$= \frac{37}{12} \text{ square units}$$

Extra Credit. [**20 points**] Choose **exactly one** of the following 3 problems.

✓ (i) Let $f(x) = 1 - x^2$. Approximate the area of the region R that lies above the x-axis, below the curve $y = f(x)$, and between the vertical lines $x = 0$ and $x = 1$, by using 4 rectangles and taking the right-endpoints of the subintervals. What is the exact area of the region R?

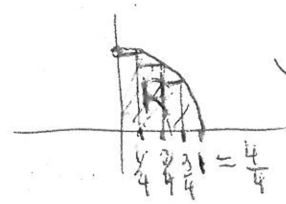
✓ (ii) Find the area of the region bounded by the curves $f(x) = 3 - x$ and $g(x) = x^2 - 9$.

✓ (iii) Find the average value of the function

$$f(x) = \begin{cases} 6, & \text{for } x < 2; \\ 3x, & \text{for } x \geq 2. \end{cases}$$

on the closed interval $[0, 4]$.

(i) $f(x) = 1 - x^2$



10 pts

Area $\approx \sum_{i=1}^{4} f\left(\frac{i}{4}\right)\cdot \underbrace{\Delta x_i}_{1/4}$

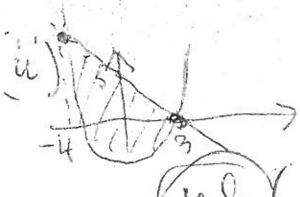
8 pts

$= f\left(\frac{1}{4}\right)\cdot\frac{1}{4} + f\left(\frac{2}{4}\right)\cdot\frac{1}{4} + f\left(\frac{3}{4}\right)\cdot\frac{1}{4} + f\left(\frac{4}{4}\right)\cdot\frac{1}{4}$

$= \left(1-\frac{1}{16}\right)\cdot\frac{1}{4} + \left(1-\frac{4}{16}\right)\cdot\frac{1}{4} + \left(1-\frac{9}{16}\right)\frac{1}{4} + (1-1)\frac{1}{4} = \frac{1}{4}\left(\frac{15}{16}+\frac{12}{16}+\frac{7}{16}\right) = \frac{34}{4\cdot 16} = \frac{17}{32}$

$\int_0^1 (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$ 2 pts

(ii)



$3 - x = x^2 - 9 \iff x^2 + x - 12 = 0, \quad x_1 = -4, \quad x_2 = 3$

10 pts

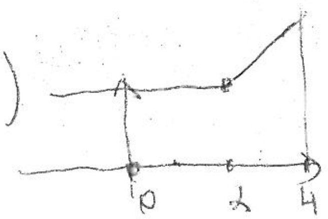
10 pts

$A = \int_{-4}^{3} (3 - x - x^2 + 9)\,dx = \left[12x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-4}^{3} = \left(12\cdot 3 - \frac{9}{2} - 9\right) - \left(-48 - 8 + \frac{64}{3}\right)$

$= 27 - \frac{9}{2} + 48 + 8 - \frac{64}{3}$

$= \cancel{83 - 9/2 + 64/3}$ sq. unit

(iii)



$f_{ave} = \frac{\int_0^2 6\,dx + 3\int_2^4 x\,dx}{4-0} = \frac{1}{4}\left[6x\right]_0^2 + \frac{3}{4}\left[\frac{x^2}{2}\right]_2^4 = 3 + \frac{\cancel{21}\ 18}{4} = \cancel{24}$

10 pts

10 pts