Happy Thanksgiving Day!

Thanks !!

Dziękuję !!

Happy Day

Szczęśliwego Dnia Dziękczynienia!

Thanksgiving

See You on December 2 !

Do zobaczenia 2-go Grudnia !

11 | 22 | 2013

**Subject:** • Reviewing:

- The Average Value Sec. 6.5
- The Arc Length Sec. 6.4

• New Application of Improper Integrals and the area —

Sec. 6.8 {
- Probability Density Function
- Probability $P(a < X < b)$
- The expected value

**Next:**

Read: More Applications Sec. 6.6 - 6.7 - 6.8

? What do we need to know?

from Chapter 6 and 5 - after the Exam II.

- Integration by Parts
- Integration using Partial Fractions
- Improper Integrals
- Area between the curves
- Volumes of the solids of revolution around different axes
- Arc Length of the curve
- The Average Value
- Applications to Physics, Engineering and Probability.

# • The Average Value of a Function ; Sec. 6.5

Example (from Larson, Hostetler + Edwards)
(LHE)

• The Speed of Sound

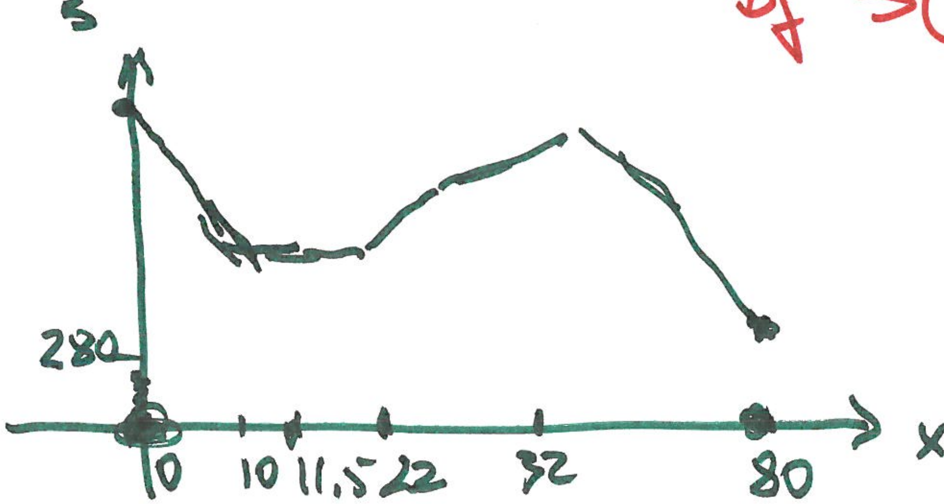
At different altitudes in earth's atmosphere, sound travels at different speeds. The speed of sound $s(x)$ (in meters per second) can be modeled by :

$$s(x) = \begin{cases} -4x + 341, & 0 \le x < 11.5 \\ 295, & 11.5 \le x < 22 \\ \frac{3}{4}x + 278.5, & 22 \le x < 32 \\ \frac{3}{2}x + 254.5, & 32 \le x < 50 \\ -\frac{3}{2}x + 404.5, & 50 \le x < 80 \end{cases}$$

where $x$ is the altitude in km.

Q: What is the average speed of sound over the interval $[0, 80]$ ?

Solution: Sketch the graph of $s(x)$



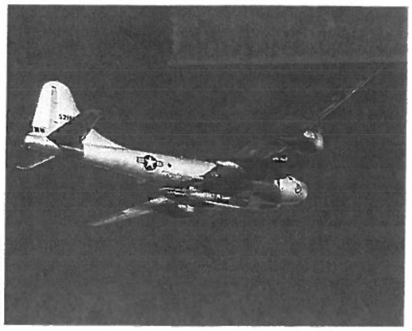

The Average Value of a Function on an Interval:

$$\bar{f}(x) = f_{ave}(x) = \frac{\int_a^b f(x)\,dx}{b-a}$$

Step 1:

Evaluate:

$$\int_0^{80} s(x)\,dx = \ldots\ldots$$



Archive Photos

The first person to fly at a speed greater than the speed of sound was Charles Yeager. On October 14, 1947, flying in an *X-1* rocket plane at an altitude of 12.8 kilometers, Yeager was clocked at 299.5 meters per second. If Yeager had been flying at an altitude under 10.375 kilometers, his speed of 299.5 meters per second would not have "broken the sound barrier." The illustration above shows the *X-1* and its *B-29* mother plane.

### EXAMPLE 5 The Speed of Sound

At different altitudes in earth's atmosphere, sound travels at different speeds. The speed of sound $s(x)$ (in meters per second) can be modeled by

$$s(x) = \begin{cases} -4x + 341, & 0 \le x < 11.5 \\ 295, & 11.5 \le x < 22 \\ \frac{3}{4}x + 278.5, & 22 \le x < 32 \\ \frac{3}{2}x + 254.5, & 32 \le x < 50 \\ -\frac{3}{2}x + 404.5, & 50 \le x < 80 \end{cases}$$

where $x$ is the altitude in kilometers (see Figure 4.34). What is the average speed of sound over the interval $[0, 80]$?

**Solution** Begin by integrating $s(x)$ over the interval $[0, 80]$. To do this, you can break the integral into five parts.

$$\int_0^{11.5} s(x)\,dx = \int_0^{11.5} (-4x + 341)\,dx = 3657$$

$$\int_{11.5}^{22} s(x)\,dx = \int_{11.5}^{22} (295)\,dx = 3097.5$$

$$\int_{22}^{32} s(x)\,dx = \int_{22}^{32} \left(\tfrac{3}{4}x + 278.5\right)dx = 2987.5$$

$$\int_{32}^{50} s(x)\,dx = \int_{32}^{50} \left(\tfrac{3}{2}x + 254.5\right)dx = 5688$$

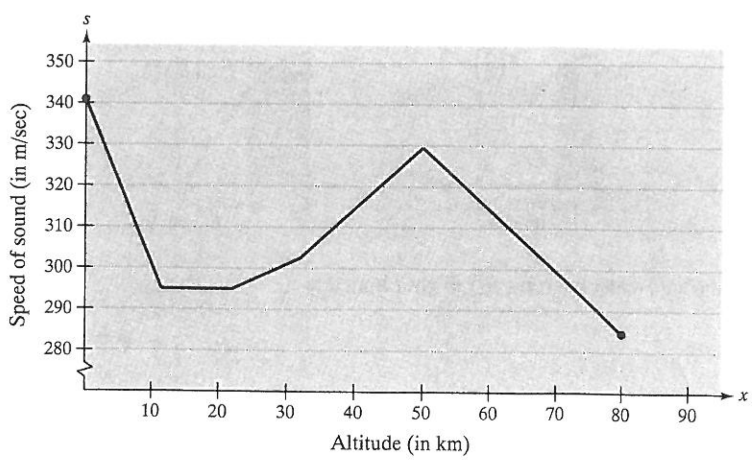
$$\int_{50}^{80} s(x)\,dx = \int_{50}^{80} \left(-\tfrac{3}{2}x + 404.5\right)dx = 9210$$

By adding the values of the five integrals, you have

$$\int_0^{80} s(x)\,dx = 24{,}640.$$

Therefore, the average speed of sound from an altitude of 0 kilometers to an altitude of 80 kilometers is

$$\text{Average speed} = \frac{1}{80}\int_0^{80} s(x)\,dx = \frac{24{,}640}{80} = 308 \text{ meters per second.}$$




Speed of sound depends on altitude.

- Arc Length

$$L = s(x) = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

continuous
($\Longrightarrow$ integrable)

where $y = f(x)$ represents
a smooth curve on $[a, b]$

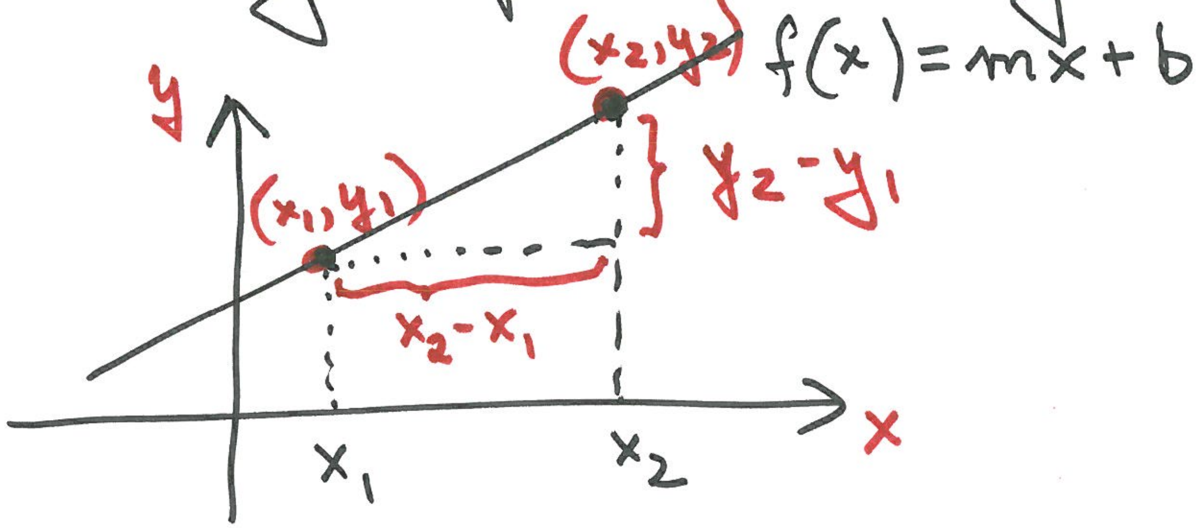
or

$$L = s(y) = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

where $x = g(y)$ represents
a smooth curve on $[c, d]$.

**Examples:** (from LHE)

The length of a Line Segment




Q: Find the arc length from $(x_1, y_1)$ to $(x_2, y_2)$ on the graph of $f(x) = mx + b$.

**Solution:** $m = f'(x) = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\Rightarrow \quad s = L = \int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2}\, dx$$

$$= \left[ \underbrace{\sqrt{\frac{(x_2-x_1)^2+(y_2-y_1)^2}{(x_2-x_1)^2}}}\ \underline{\underline{(x)}} \right]_{x_1}^{x_2}$$

$$= \sqrt{\frac{(x_2-x_1)^2+(y_2-y_1)^2}{\cancel{(x_2-x_1)^2}}} \cdot \cancel{(x_2-x_1)}$$

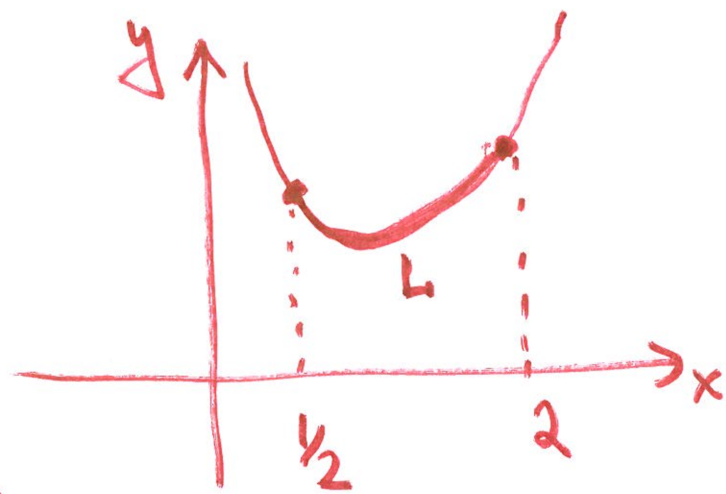
$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

which is the formula for the distance between two points in the plane.

# Examples:

- Find the arc length of $y = \frac{x^3}{6} + \frac{1}{2x}$ on $\left[\frac{1}{2}, 2\right]$

Solution:




$$\frac{dy}{dx} = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$= \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

$$L = \int_{1/2}^{2} \sqrt{1 + \frac{1}{4}\left(x^2 - \frac{1}{x^2}\right)^2}\, dx$$

$$= \int_{1/2}^{2} \sqrt{\frac{1}{4}\left(x^2 + \frac{1}{x^2}\right)^2}\, dx$$

$\Rightarrow$ see the next page for details

Observe:

$$\sqrt{1+\frac{1}{4}\left(x^4-2\cdot x^2\cdot\frac{1}{x^2}+\frac{1}{x^4}\right)}$$

$$=\sqrt{1+\frac{1}{4}x^4-\frac{1}{2}+\frac{1}{4x^4}}$$

$$=\sqrt{\frac{1}{4}x^4+\frac{1}{2}+\frac{1}{4x^4}}$$

$$=\sqrt{\frac{1}{4}\underbrace{\left(x^4+2+\frac{1}{x^4}\right)}_{\left(x^2+\frac{1}{x^2}\right)^2}}$$

$$=\sqrt{\frac{1}{4}\left(x^2+\frac{1}{x^2}\right)^2}$$

$$\Longrightarrow \int_{1/2}^{2}\sqrt{\frac{1}{4}\left(x^2+\frac{1}{x^2}\right)^2}\,dx=\int_{1/2}^{2}\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)dx \Longrightarrow \text{finish it}$$

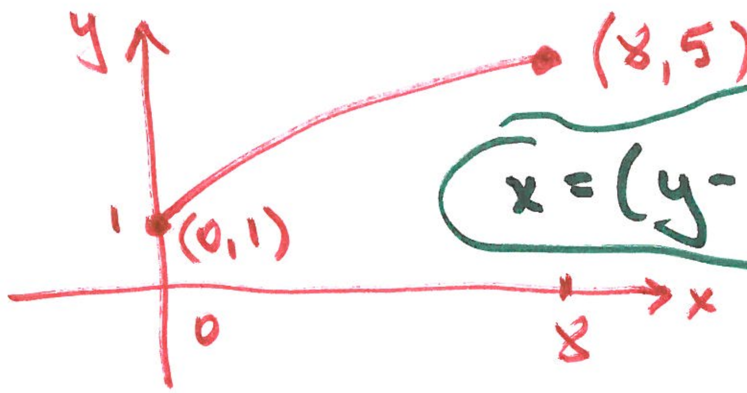
$$=\frac{1}{2}\left[\frac{x^3}{3}-\frac{1}{x}\right]_{1/2}^{2}=\frac{33}{16}$$

- Find the arc length of the graph of

$$(y-1)^3 = x^2$$

on $[0, 8]$.

Solution:




$$x = (y-1)^{3/2}$$

x-interval $[0,8]$ $\longrightarrow$ y-interval $[1,5]$

$$(y-1)^3 = 64$$

$$y-1 = 4 \Rightarrow y = 5$$

$$\boxed{\frac{dx}{dy} = \frac{3}{2}(y-1)^{\frac{1}{2}} \cdot 1}$$

$$L = \int_1^5 \sqrt{1 + \frac{9}{4}(y-1)}\, dy$$

$$= \int_1^5 \sqrt{\frac{9}{4}y - \frac{5}{4}}\, dy = \frac{1}{2}\int_1^5 (9y-5)\, dy$$

$$= \frac{1}{18}\left[\frac{(9y-5)^{3/2}}{3/2}\right]_1^5$$

$$= \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right)$$

$$\cong 9.0734$$

- Probability - Sec. 6.8

(a) • A nonnegative function $f$ is called a probability density function if:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

(b) • The probability that $x$ lies between $a$ and $b$ is given by

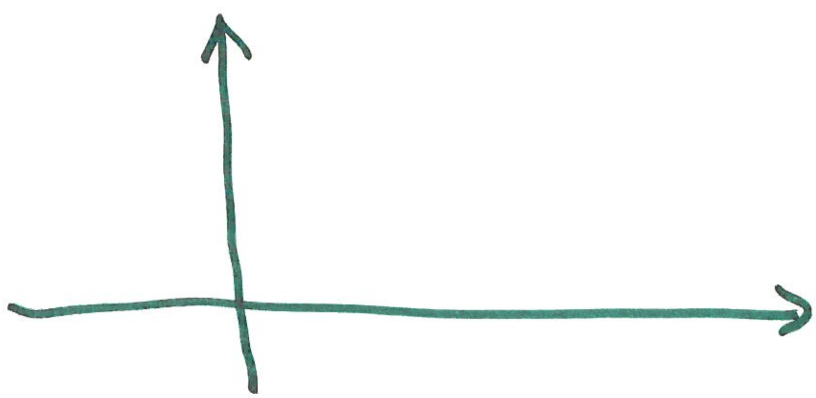
$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

(c) • The expected value of $x$ is given by

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

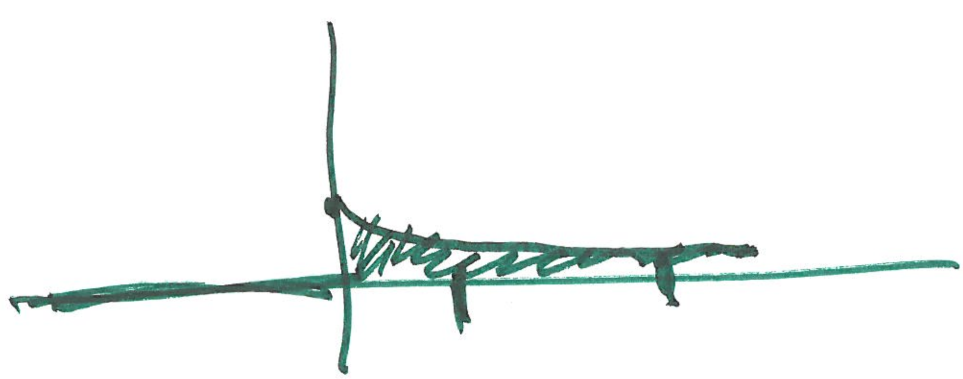
- Let

$$f(x) = \begin{cases} \frac{1}{7} e^{-\frac{1}{7}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



Checking:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad ?$$



(a)

$$\int_{-\infty}^{\infty} \frac{1}{7} e^{-1/7 x}\,dx$$

$$= \int_{-\infty}^{0} \frac{1}{7} e^{-1/7 x}\,dx + \int_{0}^{\infty} \frac{1}{7} e^{-1/7 x}\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \lim_{t \to \infty} \int_{0}^{t} \frac{1}{7} e^{-1/7 x}\,dx$$

$$= 0 + \lim_{t \to \infty} \left[ \frac{1}{7} \left(-\frac{7}{1}\right) e^{-1/7 x} \right]_0^t$$

$$= \lim_{t \to \infty} \left[ \underbrace{\frac{1}{7}(-7) e^{-1/7 t}}_{=0 \text{ when } t \to \infty} + \underbrace{e^{-1/7 \cdot (0)}}_{=1} \right]$$

$$= 1$$

(b) $P(a \le X \le b) = P(0 < x < 7) =$

$$= \int_0^7 \frac{1}{7} e^{-1/7 x}\,dx = \left[ -e^{-1/7 x} \right]_0^7$$

$$= -e^{-1/7 \cdot 7} + e^{-1/7 \cdot (0)}$$

$$= -e^{-1} + 1 = 1 - \frac{1}{e}$$

(c) Find $\int_{-\infty}^{\infty} x f(x)\,dx = EX \leftarrow$ finish it

11/20/2013

<u>Subject</u>: Applications of Integration:

- Volumes of Solids; Sec. 6.2-6.3
- Arc Length of the curve; Sec. 6.4
- The Average Value, Sec. 6.5

<u>Next</u>: More applications to engineering, physics, business, probability...

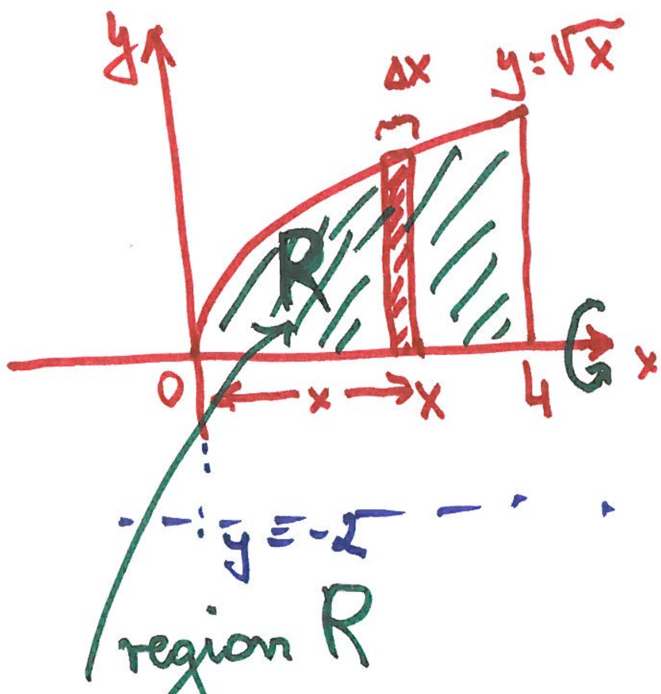
Sections: 6.6, 6.7, 6.8

## • Volumes:

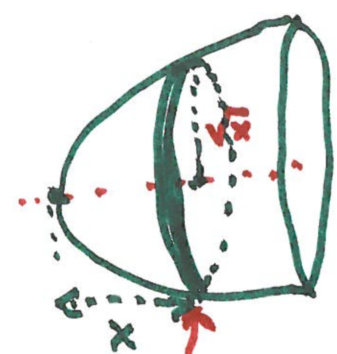
Recall the following example:

Example: Find the volume of the solid of revolution obtained by revolving the plane region R bounded by $y = \sqrt{x}$, the x-axis, and the line $x = 4$ about the x-axis.

**Solution:**




by revolving




the volume $\Delta V$ of the disk is:

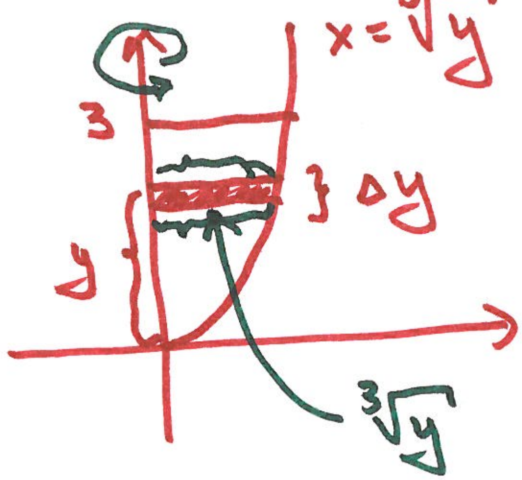
$$\Delta V = \pi (\sqrt{x})^2 \Delta x$$

$$V = \int_0^4 \pi x \, dx = 8\pi$$

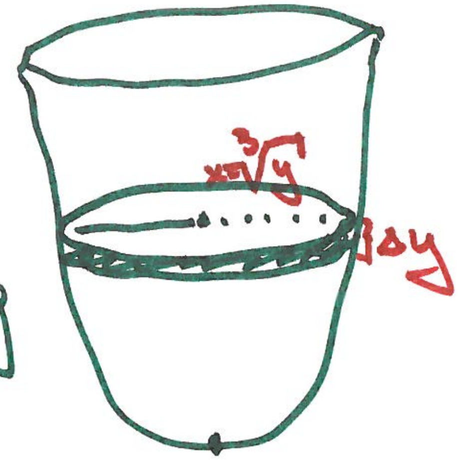
**Example:**

Find the volume of the solid generated by revolving the region bounded by the curve $y = x^3$, the $y$-axis, and the line $y = 3$ about the $y$-axis.

**Solution:**




$\xrightarrow{\hspace{3em}}$ by revolving




$$\Delta V = \pi \left(\sqrt[3]{y}\right)^2 \Delta y$$

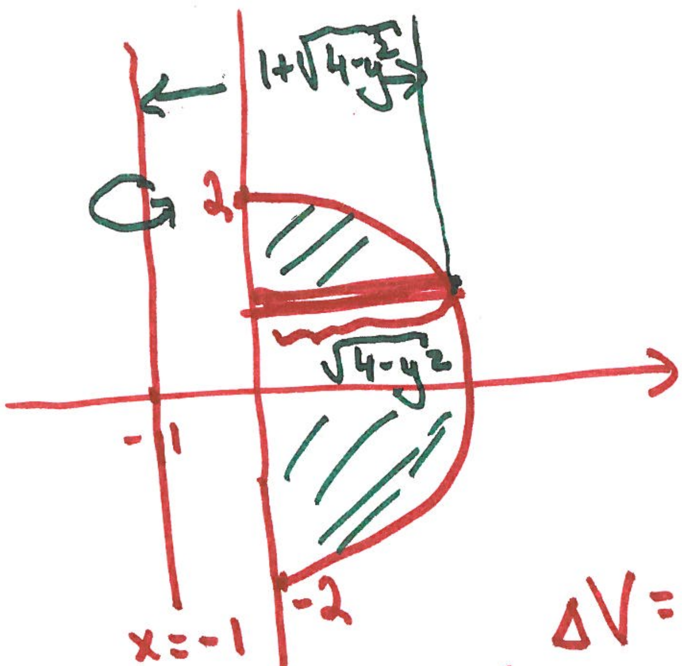
$$V = \int_0^3 \pi\, y^{2/3}\, dy = \pi \left[\frac{3}{5} y^{5/3}\right]_0^3$$

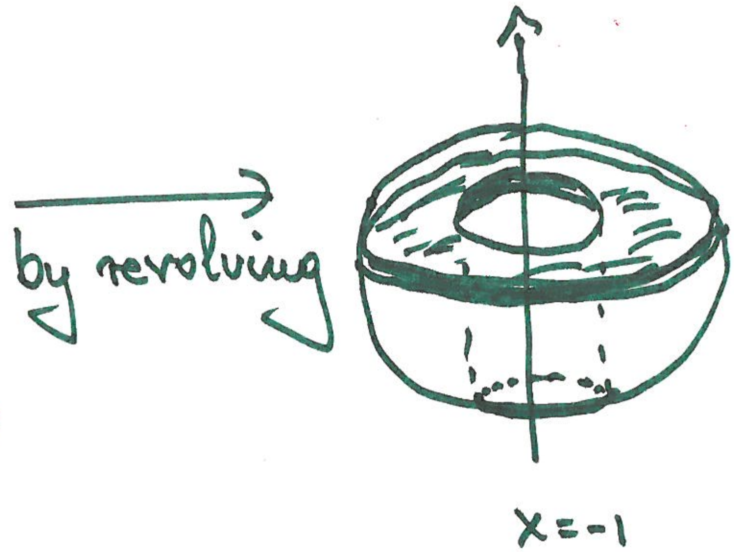
$$= \pi \frac{9\sqrt[3]{9}}{5}$$

## Example:

The semicircular region bounded by the curve $x = \sqrt{4-y^2}$ and the y-axis is revolved about the line $x = -1$.
Set up the integral that represents its volume.

## Solution:

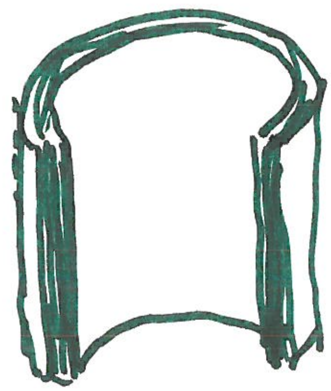


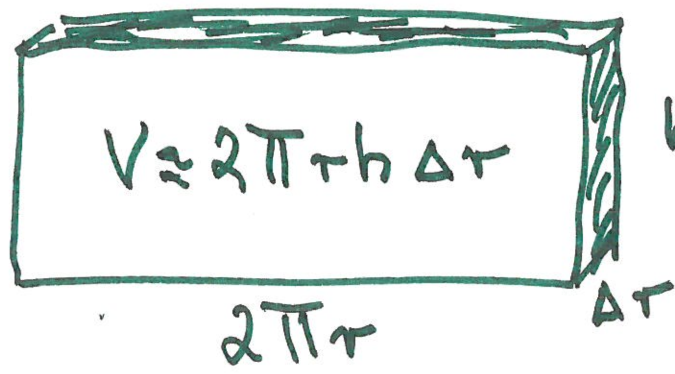




$$\Delta V = \pi\left[\left(1+\sqrt{4-y^2}\right)^2 - 1^2\right]\Delta y$$

$$V = \int_{-2}^{2} \pi\left[\left(1+\sqrt{4-y^2}\right)^2 - 1^2\right] dy$$

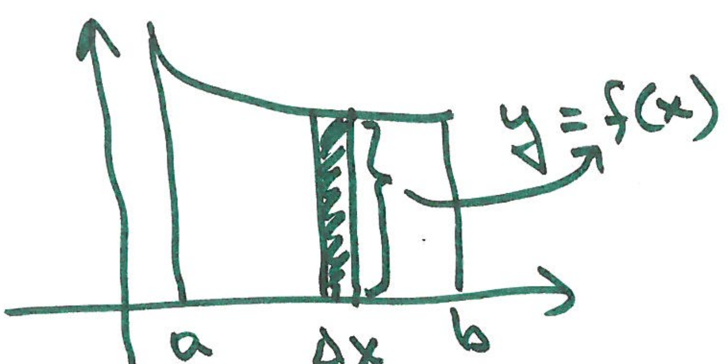
- The Method of Shells:






- $\Delta V \approx 2\pi x f(x) \Delta x$

$$\Longrightarrow V = \int_a^b 2\pi x f(x)\, dx$$




revolving about x-axis

or

$$V = \int_c^d 2\pi y f(y)\, dy$$

if revolving about the y-axis

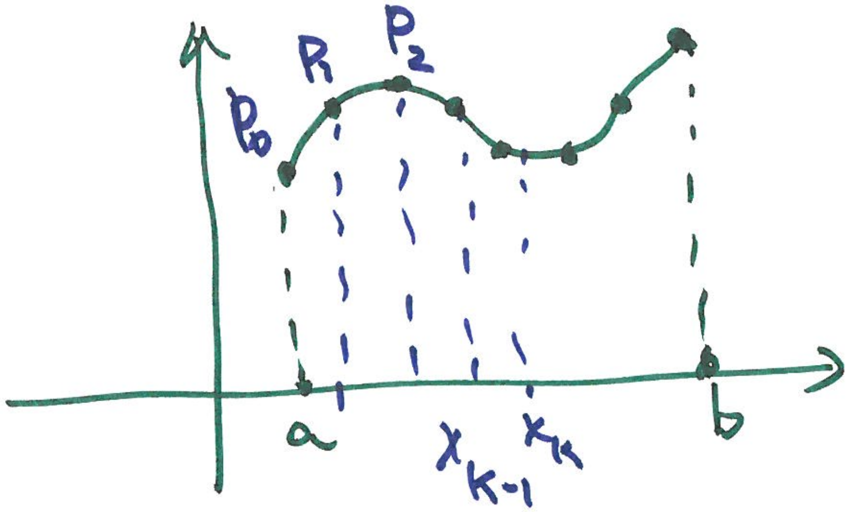
- Arc Length

Arc Length Problem:

Suppose that $y = f(x)$ is a smooth curve on the interval $[a, b]$.

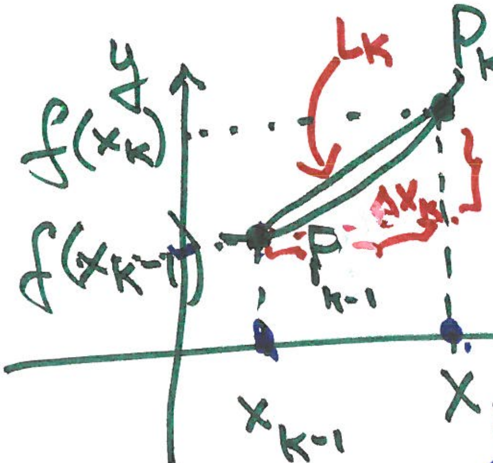
$f'$ is continuous

Define and a find a formula for the length of a plane curve $y = f(x)$ over the interval $[a, b]$.




- breaking the curve into small segments
- approximate the curve segments by line segments
- add them $\Rightarrow$ RS

$$\boxed{\Delta x_k = x_k - x_{k-1}}$$




$$\Delta y_k = f(x_k) - f(x_{k-1})$$

$$\boxed{L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}}$$

$$= \sqrt{\underline{(\Delta x_k)^2} + \underbrace{[f(x_k) - f(x_{k-1})]^2}_{\Delta f(x_k)}}$$

$$L \approx \sum_{k=1}^{n} L_k$$

Observe:

$$\frac{\Delta f(x_k)}{\Delta x_k} = \boxed{\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*)}$$

$$\Rightarrow f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k \qquad \text{point} \in (x_{k-1}, x_k);$$

$$\Longrightarrow \quad L \approx \underbrace{\sum_{k=1}^{n} \sqrt{1+\left[f'(x_k^*)\right]^2}\,\Delta x_k}$$

$$\Longrightarrow$$

$$L = \lim_{\max \Delta x_k \to 0} \Big( \qquad \Big)$$

$$L := \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$$

or in different form:

$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad (1)$$

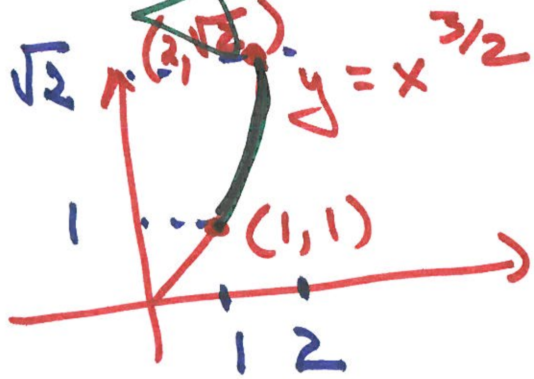
or

$$L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad (2)$$

**Example:**

Find the arc length of the curve $y = x^{3/2}$ from $(1,1)$ to $(2,\sqrt{2})$ in both ways.

**Solution:**




$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

from (1) $\quad L = \int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx$

u - substitution

$u = 1 + \frac{9}{4}x \implies du = \frac{9}{4} dx$

and then change the limits of integration to u-limits:

$u = \frac{13}{4}, \quad u = \frac{22}{4}$

$$L = 4/9 \int_{13/4}^{22/4} u^{1/2}\, du = \frac{8}{27} u^{3/2} \Big]_{13/4}^{22/4} = \ldots$$

$$\int \sqrt{1+\frac{9}{4}x}\,dx = \frac{4}{9}\int u^{1/2}\,du$$

$$= \frac{8}{27}u^{3/2} = \boxed{\frac{4}{9}\left(1+\frac{9}{4}x\right)^{3/2}}$$

$$\Rightarrow L = \frac{4}{9}\left[\left(1+\frac{9}{4}x\right)^{3/2}\right]_1^2$$

$$= \frac{4}{9}\left[\left(1+\frac{9}{4}\cdot 2\right)^{3/2} - \left(1+\frac{9}{4}\cdot 1\right)^{3/2}\right]$$

$$= \frac{4}{9}\left[\left(\frac{11}{2}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

$$= \frac{4}{9}\left[\frac{22}{4}\cdot\sqrt{\frac{22}{4}} - \frac{13}{4}\sqrt{\frac{13}{4}}\right]$$

$$= \frac{22\sqrt{22} - 13\sqrt{13}}{27} \approx 2.09$$

To apply formula (2) we rewrite the equation

$$\boxed{y = x^{3/2}} \Longrightarrow x = y^{2/3}$$

$$\Longrightarrow \frac{dx}{dy} = \frac{2}{3}\, y^{-1/3}$$

$$\Longrightarrow L = \int_1^{2\sqrt{2}} \sqrt{1 + \frac{4}{9} y^{-2/3}}\; dy$$

$$= \int_1^{2\sqrt{2}} y^{-1/3} \sqrt{\underbrace{9y^{2/3} + 4}_{=u}}\; dy$$

$$\left.\begin{array}{l} u = 9y^{2/3} + 4 \\ \Longrightarrow du = 6y^{-1/3}\, dy \end{array}\right) \Longrightarrow \text{finish it}$$

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**Subject:**

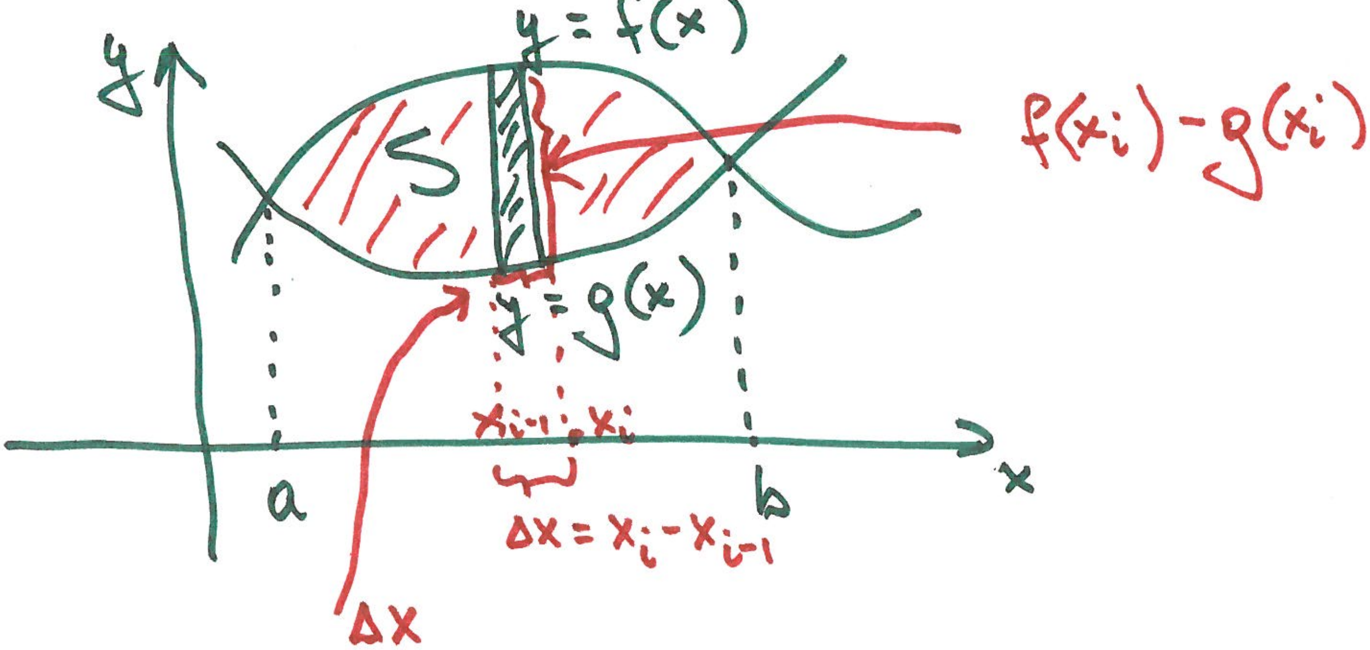
Applications of Integrals

- Area between curves
  Sec. 6.1
- Volumes
  Sec. 6.2 and 6.3

**Next:**

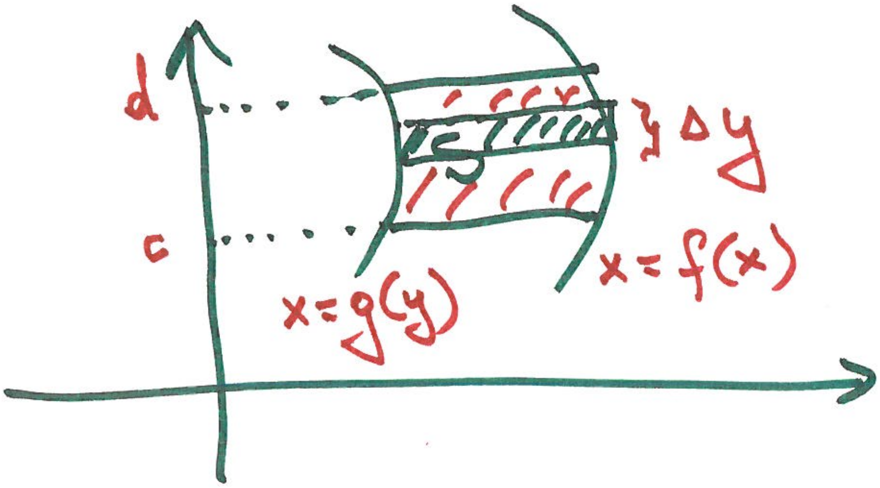
- Arc Length , Sec. 6.4
- Average Value of a Function
  Sec. 6.5

- Areas between curves:




$$\text{Area}(S) = \int_a^b [f(x) - g(x)]\, dx$$

continuous functions
and $f(x) \geq g(x)$
for all $x$ in $[a, b]$.




$$A = \int_c^d [f(y) - g(y)]\,dy$$

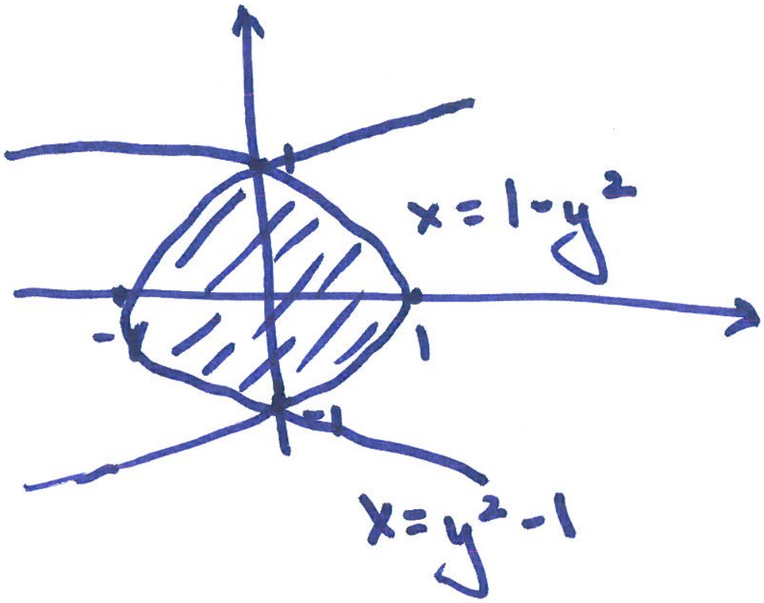
Examples: Your HW is: 7, 8, 11, 17 from Sec. 6.1

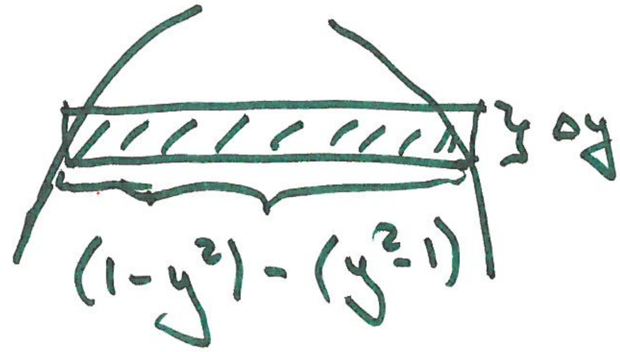
Let us do similar problems:

#9: $x = 1 - y^2$ and $x = y^2 - 1$

Solution:

- Sketch the region enclosed by the curves
- Draw by a typical approximating rectangle




$$1 - y^2 = y^2 - 1$$

$$\iff \quad 2y^2 = 2$$

$$\iff \quad y^2 = 1$$

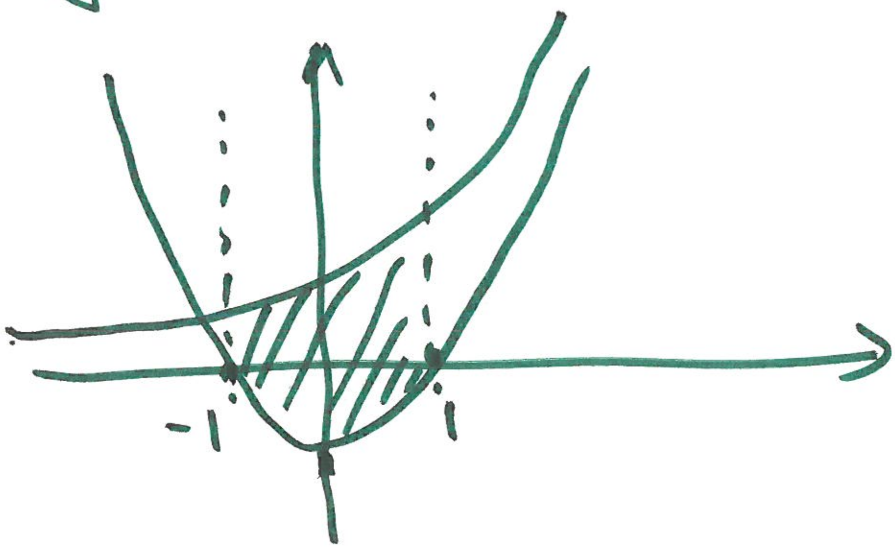
$$\iff \quad y = \pm 1$$

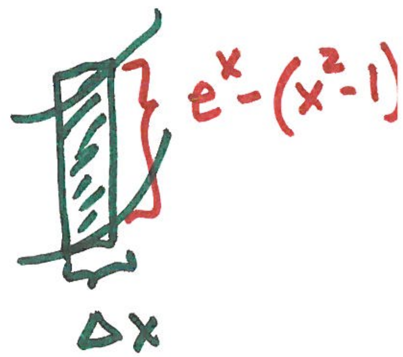
$$A = \int_{-1}^{1} \left[ (1-y^2) - (y^2-1) \right] dy$$

$$= \int_{-1}^{1} 2(1-y^2)\, dy = 2 \cdot \int_{0}^{1} 2(1-y^2)\, dy$$

$$= 4 \left[ y - \frac{1}{3} y^3 \right]_0^1 = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$$

• $y = e^x$, $y = x^2 - 1$, $x = -1$, $x = 1$



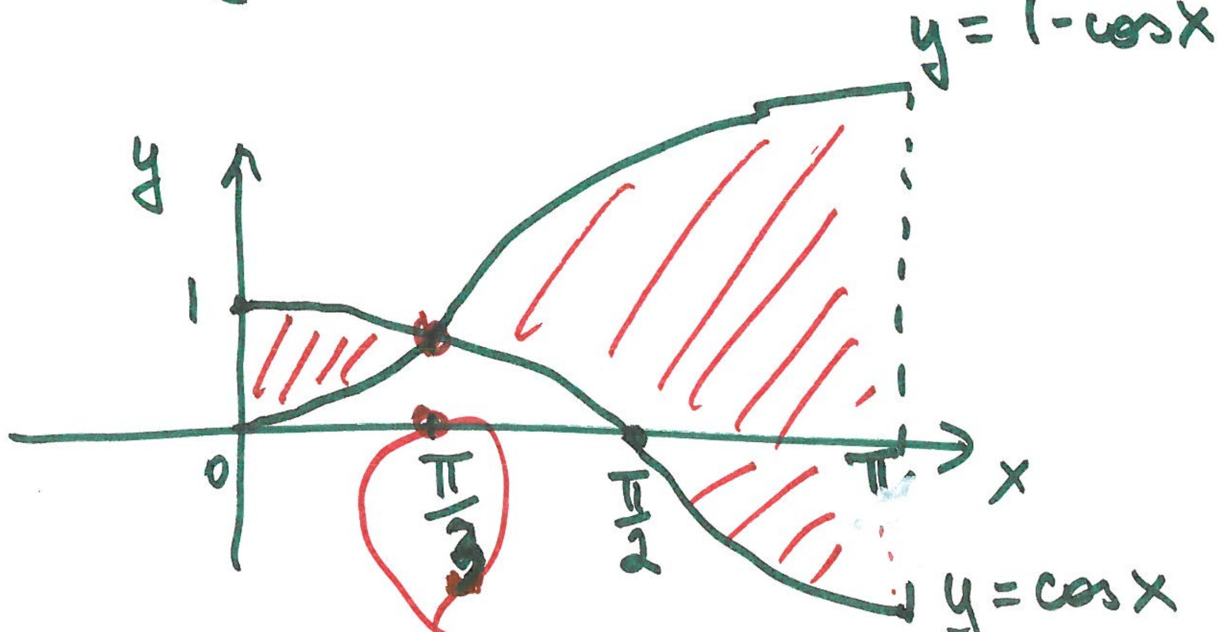




$$A = \int_{-1}^{1} \left[ e^x - (x^2 - 1) \right] dx = \left[ e^x - \frac{1}{3}x^3 + x \right]_{-1}^{1}$$

$$= \left(e - \frac{1}{3} + 1\right) - \left(e^{-1} + \frac{1}{3} - 1\right)$$

$$= e - \frac{1}{e} + \frac{4}{3}$$

\# 24. $\begin{cases} y = \cos x \text{ and } y = 1 - \cos x \\ 0 \leq x \leq \pi \end{cases}$




$\cos x = 1 - \cos x$ on $[0, \pi]$

$\iff 2\cos x = 1 \iff \cos x = \frac{1}{2}$

$\iff x = \frac{\pi}{3}$

$$A = \int_0^{\pi/3} [\cos x - (1 - \cos x)]\,dx$$

$$+ \int_{\pi/3}^{\pi} [(1 - \cos x) - \cos x]\,dx$$

$$= \Big[2\sin x - x\Big]_0^{\pi/3} + \Big[x - 2\sin x\Big]_{\pi/3}^{\pi} = \underline{2\sqrt{3} + \frac{\pi}{3}}$$

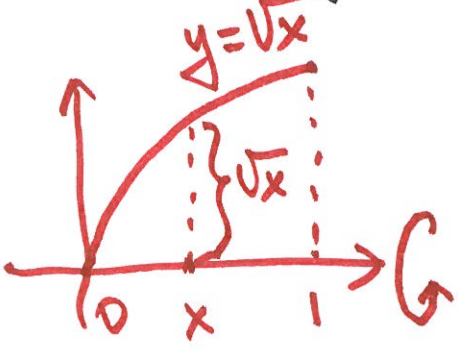
- Volumes

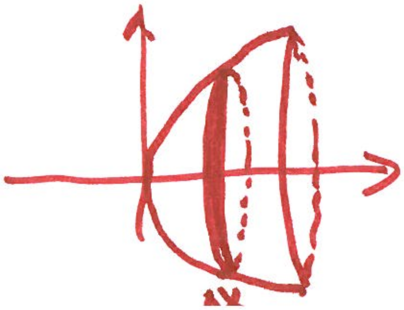
$$V = \int_a^b A(x)\,dx$$

cross-sectional area of S, in the plane

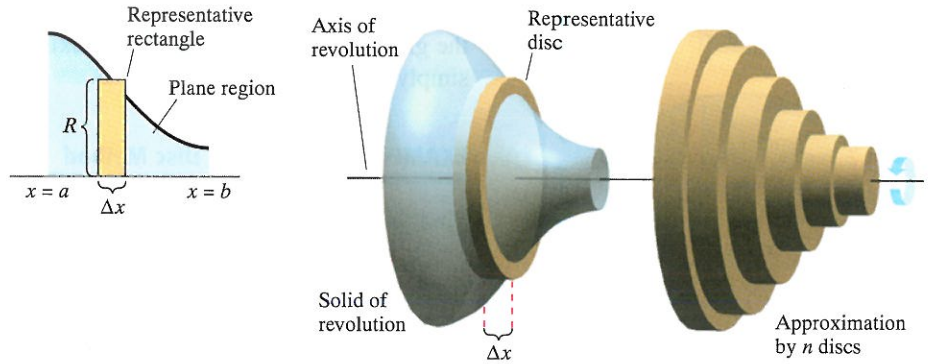
- Disk method

<u>Example</u>: Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y=\sqrt{x}$ from 0 to 1.



From : Larson, Hostetler and Edwards

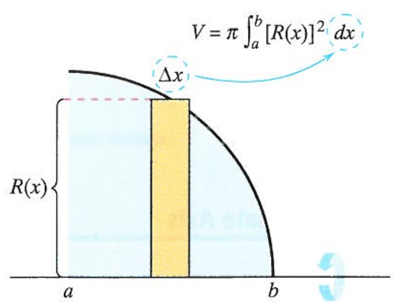



Disc method
**Figure 6.15**

This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. Therefore, you can define the volume of the solid as

$$\text{Volume of solid} = \lim_{\|\Delta\| \to 0} \pi \sum_{i=1}^{n} [R(x_i)]^2 \, \Delta x$$

$$= \pi \int_a^b [R(x)]^2 \, dx.$$

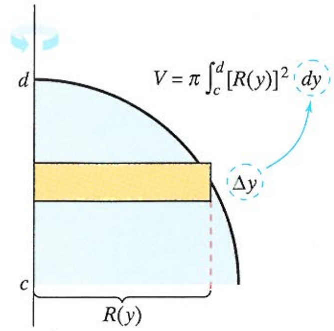



Horizontal axis of revolution

Schematically, the disc method looks like this.

| *Known Precalculus Formula* | | *Representative Element* | | *New Integration Formula* |
|---|---|---|---|---|
| Volume of disc $V = \pi R^2 w$ | ⇨ | $\Delta V = \pi [R(x_i)]^2 \Delta x$ | ⇨ | Solid of revolution $V = \pi \int_a^b [R(x)]^2 \, dx$ |

A similar formula can be derived if the axis of revolution is vertical.




Vertical axis of revolution

**The Disc Method**

To find the volume of a solid of revolution with the **disc method,** use one of the following, as indicated in Figure 6.16.

*Horizontal Axis of Revolution*

$$\text{Volume} = V = \pi \int_a^b [R(x)]^2 \, dx$$

*Vertical Axis of Revolution*

$$\text{Volume} = V = \pi \int_c^d [R(y)]^2 \, dy$$

Solution:

The area of the cross-section is

$$A = \pi\left(\sqrt{x}\right)^2 = \pi x$$

$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \pi x \, dx$$

$$= \pi \frac{x^2}{2}\Big]_0^1 = \frac{\pi}{2}$$

**Example**: Rotating about $y$-axis

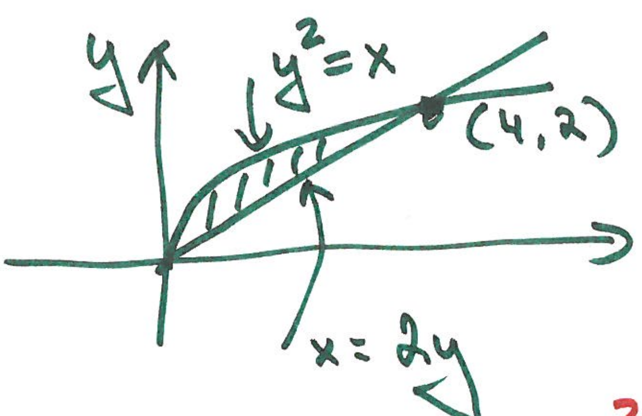
$$\boxed{V = \int_c^d A(y)\, dy}$$

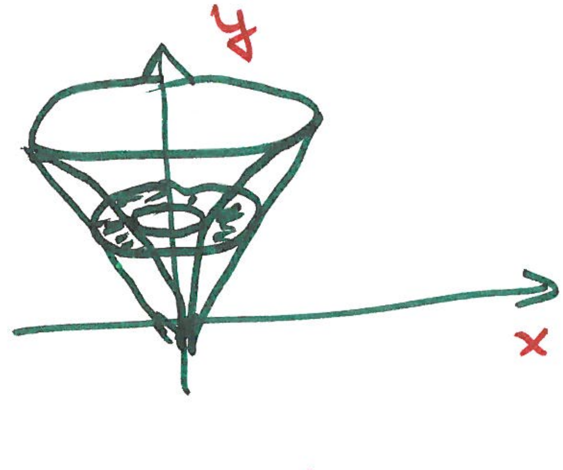
HW: Sec. 6.2: 4, 10, 13, 16

Let us do: from Stewart:

$$\boxed{\#7:\ y^2 = x,\ x = 2y \text{ about } y\text{-axis}}$$

Solution:




$$\left.\begin{array}{ll} \text{inner radius}: & y^2 \\ \text{outer radius}: & 2y \end{array}\right) \Longrightarrow A = \pi\,(\text{outer radius})^2 - \pi\,(\text{inner radius})^2$$

$$A = \pi\left[(r_{out})^2 - (r_{in})^2\right].$$

$$A(y) = \pi(2y)^2 - \pi(y^2)^2$$

$$= \pi\left[4y^2 - y^4\right]$$

$$V = \int_0^2 A(y)\,dy$$

$$= \pi\int_0^2 (4y^2 - y^4)\,dy$$

$$= \pi\left[\frac{4y^3}{3} - \frac{y^5}{5}\right]_0^2$$

$$= \pi\left(\frac{32}{3} - \frac{32}{5}\right)$$

$$= \frac{64}{15}\pi \quad (\text{units})^3$$

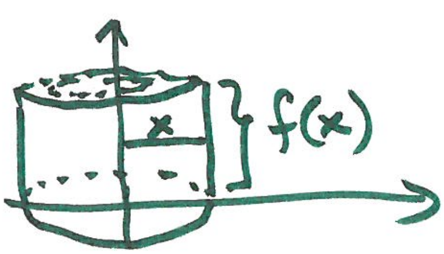
- Volumes by Cylindrical Shells

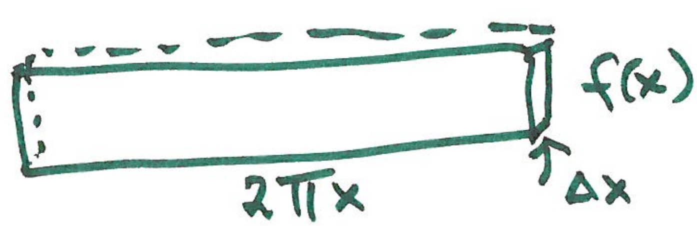
$$V = (\text{circumference})(\text{height})(\text{thickness})$$

$$= \underbrace{2\pi r}_{\text{circumference}}\ \underbrace{h}_{\text{height of shell}} \cdot \underbrace{\Delta r}_{\text{thickness of the shell}}$$

$$V = \int_a^b 2\pi x\, f(x)\, dx$$

$$0 \le a < b$$

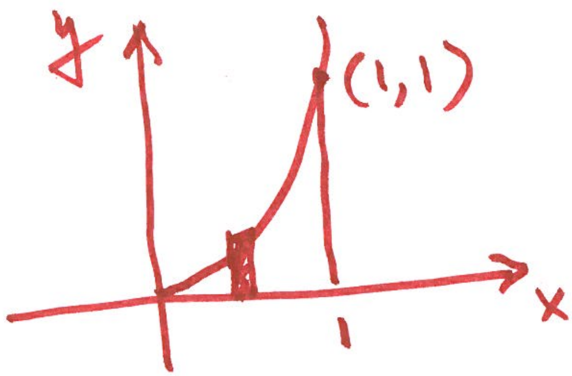


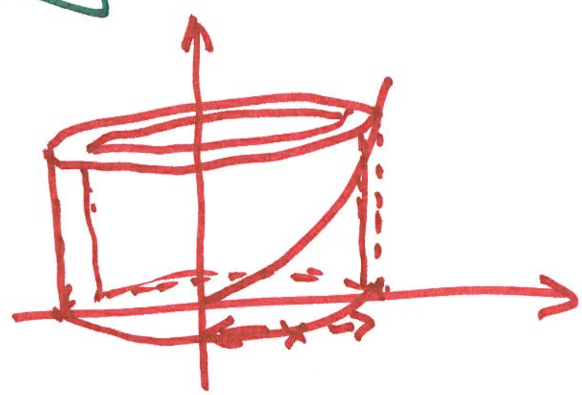



HW: Sec. 6.3 : 3, 5, 7, 9

Let us do : from Stewart:

#4 : Use the method of cylindrical shells to find the volume generated by the region bounded by $y = x^2$, $y = 0$, $x = 1$ about the $y$-axis.




$$V = \int_0^1 2\pi x \cdot x^2 \, dx = 2\pi \int_0^1 x^3 \, dx$$

$$= 2\pi \left[ \frac{1}{4} x^4 \right]_0^1 = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$

11/15/2013

**Subject**: • Review Chapter 5
What we need to know?

• Revisit Sec. 6.1
Areas between curves

**Next**: • Area between curves; 6.1
• Volumes ; 6.2 and 6.3
• Arclength ; 6.4
• The Average Value ; 6.5

Thanksgiving Week Reading:
Applications of Integrals
6.6 - 6.7 - 6.8

- Review of Chapter 5 – Most Important Concepts

- Definite Integral as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

lies in the i-th subinterval $[x_{i-1}, x_i]$

$x_i^*$ can be left endpoint
right endpoint
or middle point / midpoint

$\Delta x = \frac{b-a}{n}$

If the limit exists, we say that f is integrable on $[a,b]$,
and if is integrable on $[a,b]$ then

in particular: $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

right endpoints: $x_i = a + i\Delta x$

- Evaluation Theorem.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

f is continuous on [a, b]

F is any antiderivative

$F'(x) = f(x)$

Example:

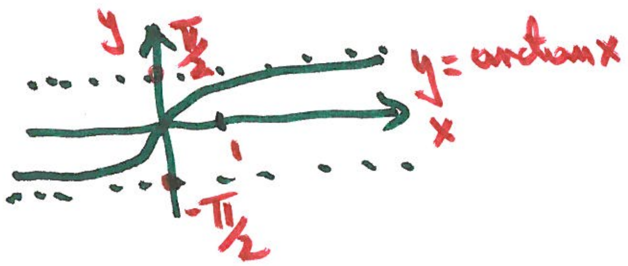
- $\displaystyle\int_1^2 e^x\,dx = e^x\Big]_1^2 = e^2 - e$

- $\displaystyle\int_0^{\pi/2} \cos x\,dx = \sin x\Big]_0^{\pi/2} = 1 - 0 = 1$

- $\displaystyle\int_0^1 \frac{3}{x^2+1}\,dx = 3\tan^{-1}x\Big]_0^1$

$$= 3\Big[\underbrace{\tan^{-1}1}_{\tan \pi/4 = 1} - \tan^{-1}0\Big]$$

$$= 3\Big[\pi/4 - 0\Big] = \frac{3}{4}\pi$$



- The Fundamental Theorem of Calculus

$$g(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$$

with $\quad g'(x) = f(x)$ for $a < x < b$.

Example:

- $\frac{d}{dx}\int_0^{x} \sqrt{1+t^2}\,dt = \sqrt{1+x^2}$

- $\frac{d}{dx}\int_0^{x^2} \sqrt{1+t^2}\,dt = \sqrt{1+(x^2)^2}\cdot(x^2)'$

$$= \sqrt{1+x^4}\cdot 2x$$

- $\frac{d}{dx}\int_0^{\cos x} \sqrt{1+t^2}\,dt = \sqrt{1+(\cos x)^2}\cdot(\cos x)'$

$$= \sqrt{1+\cos^2 x}\cdot(-\sin x)$$

- Integration Methods:

The Substitution Rule:

$$\int x\,(x^2+5)^{10}\,dx = \frac{1}{2}\int u^{10}\,du =$$

Let $u = x^2+5$

$du = 2x\,dx$

$\Rightarrow x\,dx = \frac{1}{2}\,du$

$$= \frac{1}{2}\,\frac{u^{11}}{11} = \frac{1}{22}(x^2+5)^{11} + C$$

Checking:

$$\left[\frac{1}{22}(x^2+5)^{11}+C\right]' = \frac{11}{22}\cdot(x^2+5)^{10}\cdot 2x$$

$$= x\,(x^2+5)^{10}$$

- Integration by Parts:

Examples:

- $\int \underbrace{r}_{u} \underbrace{e^{r/2}\, dr}_{dv}$

$u = r$ $\qquad dv = e^{r/2}\, dr$

$du = dr$ $\qquad v = 2e^{r/2}$

$$\Rightarrow \int r e^{r/2}\, dr = r \cdot 2e^{r/2} - \int 2e^{r/2}\, dr$$
$$= 2re^{r/2} - 4e^{r/2} + C$$
$$= \boxed{2(r-2)e^{r/2} + C}$$

Check it!

- $\int t \sin 2t \, dt$

$u = t, \qquad dv = \sin 2t \, dt$

$du = dt \qquad v = -\frac{1}{2}\cos 2t$

$$\Rightarrow \int t \sin 2t \, dt = -\frac{1}{2} t \cos 2t + \frac{1}{2}\int \cos 2t \, dt$$

Check it! $= -\frac{1}{2} t \cos 2t + \frac{1}{4} \sin 2t + C$

- Integration by using Partial Fractions:

Recall:
$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+c$$

Examples:

- $$\int \frac{5x+1}{2x^2-x+1}\,dx$$

$$\frac{5x+1}{2x^2-x\ast 1} = \frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

Multiplying both sides by $(2x+1)(x-1)$ to get $5x+1$

$$A(x-1)+B(2x+1) = 5x+1$$

$$(A+2B)x + (-A+B) = 5x+1$$

$$\Rightarrow \begin{array}{r} A+2B=5 \\ -A+B=1 \\ \hline 3B=6 \end{array} \Rightarrow B=2 \qquad A=1$$

$$\Rightarrow \int \frac{5x+1}{2x^2-x-1}\,dx = \int \frac{1}{2x+1}\,dx + \int \frac{2}{x-1}\,dx = \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$$

# Integration Using Tables:

- $\int e^{2x} \arctan(e^x)\, dx =$

Substitution first:

$u = e^x$

$du = e^x dx \quad \Rightarrow e^x \underbrace{dx}_{} = u \, \underset{= du}{dx} =$

$\Rightarrow dx = \frac{du}{u}$

$\int u^2 \arctan u \left(\frac{du}{u}\right)$

$= \int u \arctan u \, du =$ from the table (92)

$= \frac{u^2+1}{2} \arctan u - \frac{u}{2} + C$

$= \frac{1}{2}\left(e^{2x}+1\right) \arctan(e^x) - \frac{1}{2} e^x + C$

Check it!

- Approximation Methods:


$$\int_0^3 \frac{dt}{1+t^2+t^4}, \quad n = 6$$

$$\Delta t = \frac{3-0}{6} = \frac{1}{2}$$

$$T_6 = \frac{1}{2\cdot 2}\left[f(0) + 2f\left(\tfrac{1}{2}\right) + 2f(1) + 2f\left(\tfrac{3}{2}\right) + 2f(2) + 2f\left(\tfrac{5}{2}\right) + f(3)\right]$$

$$\approx \underline{\underline{0.895122}}$$

$$M_6 = \frac{1}{2}\left[f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right) + f\left(\tfrac{9}{4}\right) + f\left(\tfrac{11}{4}\right)\right]$$

$$\approx \underline{\underline{0.895478}}$$

$$S_6 = \frac{1}{2\cdot 3}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + 2f(2) + 4f\left(\tfrac{5}{2}\right) + f(3)\right]$$

$$\approx \underline{\underline{0.898014}}$$

# Improper Integrals:

- $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^3}\,dx$

$$= \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^3}\,dx$$

$$\int \frac{1}{x(\ln x)^3}\,dx = \int \frac{1}{u^3}\,du = \int u^{-3}\,du$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \frac{u^{-2}}{-2} = -\frac{1}{2}\,\frac{1}{\ln^2 x}$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2}\,\frac{1}{\ln^2 x}\right]_e^t = \lim_{t\to\infty}\left[-\frac{1}{2}\,\frac{1}{\ln^2 t} + \frac{1}{2}\,\frac{1}{\ln^2 e}\right]$$

$$= \frac{1}{2\cdot 1} = \frac{1}{2}$$

0

The integral converges.

11/13/2013

**Subject**: • Integration Method: Numerical Approximation
• Improper Integrals

Next time:
• Review Sections: 5.6 - 5.10
• Revisit: The Area Between Curves; Sec. 6.1

Motivation:

Evaluate: $\int_0^1 \sin(x^2)\,dx$ ?

How to do it? ?

substitution? What kind of?

Numerical integration is the way to go!

Approximations are all you can hope for:

- The rectangle rule or midpoint rule; using rectangales
- Trapezoid rule; using trapezoids
- Simpson's rule - using pieces of parabolas

Computers seem to be better at this sort of thing than people. Try using MATLAB

see: Learning MATLAB by Tobin A. Driscoll SIAM publication

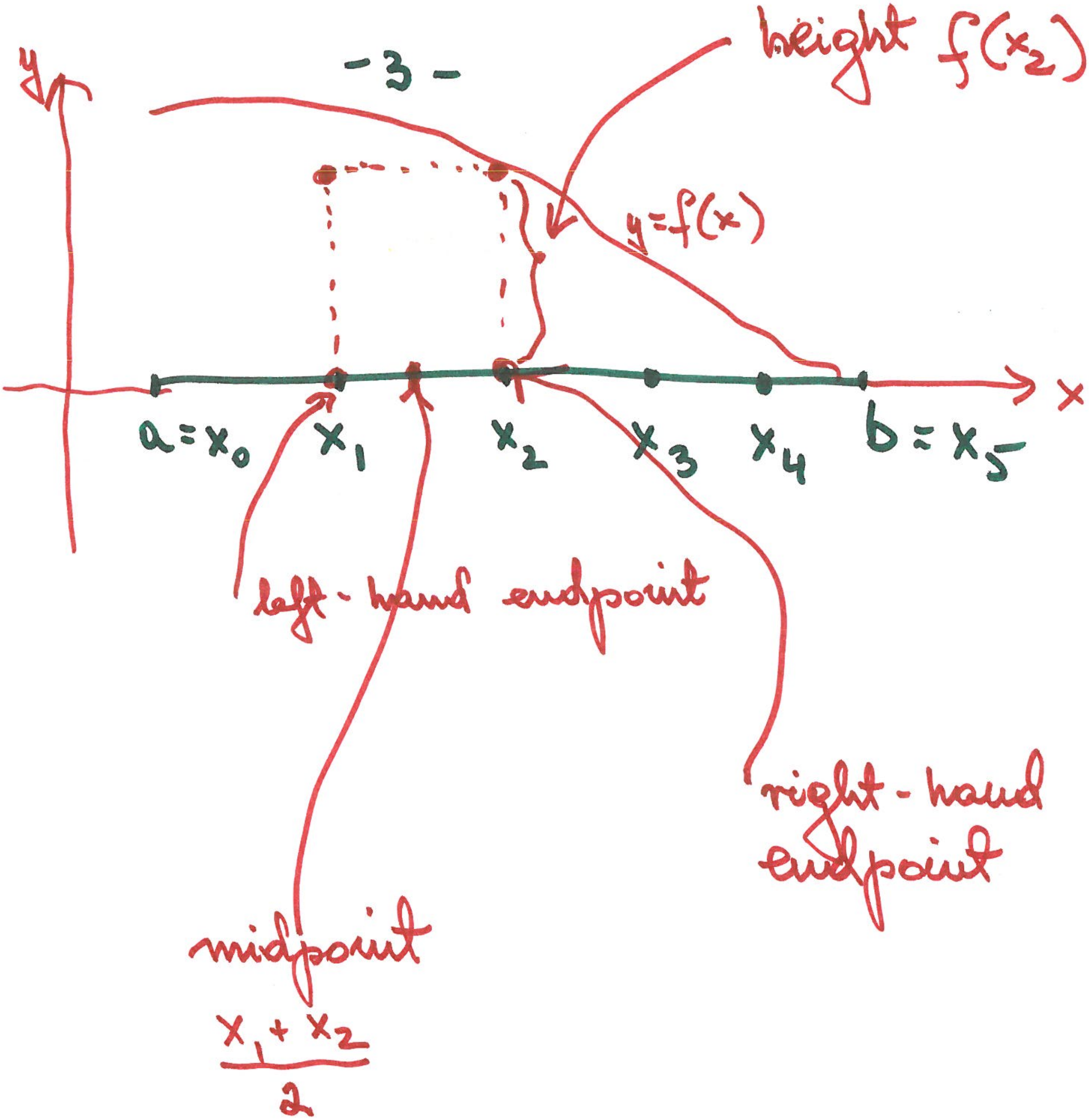
$\boxed{\text{Computers have these types of methods built into them.}}$

Start any approximation of:

$$\int_a^b f(x)\,dx$$

with dividing the interval $[a, b]$ into $n$ equal pieces, called subintervals with length

$$\frac{b-a}{n}$$


height $f(x_2)$
y
$y=f(x)$
x
$a=x_0$
$x_1$
$x_2$
$x_3$
$x_4$
$b=x_5$
left-hand endpoint
right-hand endpoint
midpoint
$\frac{x_1+x_2}{2}$

 from Collin Adams

You can extend it to an arbitrary n

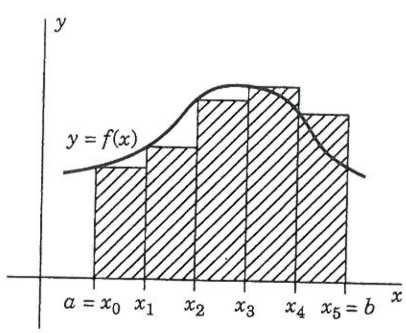
**Rectangle Rule (Using Left-Hand Endpoints)**

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}[f(x_0)+f(x_1)+\cdots+f(x_{n-1})]$$

Notice you should stop at $x_{n-1}$, not $x_n$.

If we use the right-hand endpoints instead, we get a slightly different rectangle rule.

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}[f(x_1)+f(x_2)+\ldots+f(x_n)]$$




Approximating the definite integral $\int_a^b f(x)\,dx$ with rectangles.

If we choose rectangles whose height is given by the value of the function at the center of the rectangle, instead of the left-hand endpoint, we get an approximation of the integral called the midpoint rule

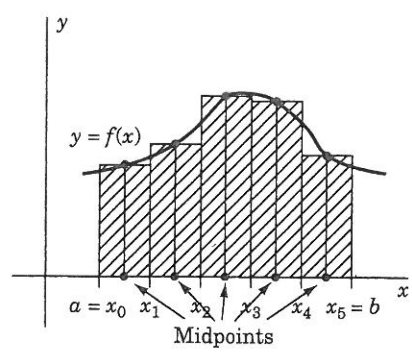
**Midpoint Rule**

$$\boxed{\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left[f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{n-1}+x_n}{2}\right)\right]}$$

where

$$\frac{x_0+x_1}{2},\qquad \frac{x_1+x_2}{2},\qquad \ldots,\qquad \frac{x_{n-1}+x_n}{2}$$

are the midpoints of the $n$ equal intervals between $a = x_0$ and $b = x_n$.




Midpoint rule.

Illustrate it with $\int_1^2 \frac{1}{x}\,dx$, $n=5$

see p. 403 of Stewart

**Trapezoid Rule**

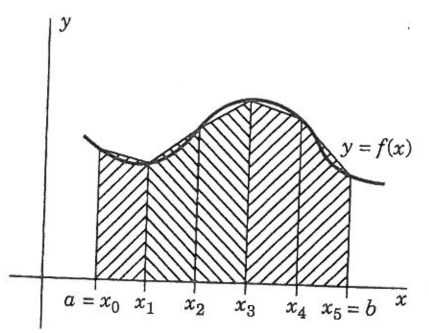
$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left[\frac{f(x_0)}{2} + f(x_1) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right]$$

Here you include both $f(x_0)$ and $f(x_n)$, but only half of each. This comes out of the fact that the area of a trapezoid with left height $L$, right height $R$, and width $W$ is $\dfrac{(L+R)W}{2}$.

If instead we replace bits of $f(x)$ with pieces of parabolas that look even more like $f(x)$, we get Simpson's rule.

**Simpson's Rule**

$$\int_a^b f(x)\,dx$$
$$\approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$




See example 1 p. 403

$\int_1^2 \frac{1}{x}\,dx$, $n = 5$

# Improper Integrals:

## Sec. 5.10

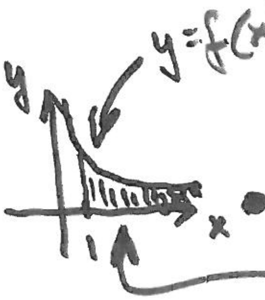
$$\int_{+a}^{+\infty} f(x)\,dx \;; \qquad \int_{-\infty}^{b} f(x)\,dx \,,$$

$$\int_{-\infty}^{\infty} f(x)\,dx$$

of the first type with limit (s) of integration being $+\infty$ or $-\infty$.

• Example: Evaluate: $\int_1^\infty \frac{1}{x^2}\,dx$

Think of $\infty$ here as the limit of




numbers which are getting very large, or $\lim_{b\to\infty} b$. So the official interpretation of our improper integral is

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left.\frac{-1}{x}\right|_1^b = \lim_{b\to\infty}\left(\left(\frac{-1}{b}\right)-(-1)\right) = 0-(-1) = 1$$

So in this case, that area actually turned out to be 1.

We will always interpret an infinite limit improper integral this way:

Definition

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx}$$

Sometimes an improper integral gives a finite number, as happened above. Then we say the improper integral *converges.* But sometimes the limit is $\infty$ or doesn't exist. Then we say the improper integral *diverges.*

**Example** Find $\int_1^\infty \frac{1}{x}\,dx$. $\Rightarrow$ $\int_1^\infty \frac{1}{x}\,dx$ diverges

**Solution** This doesn't look very different from the previous example, but whammo, when we take a limit to compute it, we get

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} (\ln x)\Big|_1^b = \lim_{b\to\infty}((\ln b)-(0)) = \infty$$

This one *diverges.*

We also define

$$\boxed{\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx}$$

and

$$\boxed{\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx}$$

for any real number $c$ that we want to use.

# The Second Type of Improper Integrals.

The integrand is undefined at the point $\in [a, b]$;

either at endpoints of the interval or at a point inside of the interval of integration:

$$\int_0^2 \frac{1}{x}\, dx \quad , \quad \int_{-2}^0 \frac{1}{x}\, dx$$

or $$\int_{-2}^2 \frac{1}{x}\, dx$$

Let us compute

$$\int_{-1}^{2} \frac{1}{x^{2/3}}\, dx$$

Here $f(x) = \frac{1}{x^{2/3}}$ is undefined at $x = 0 \Rightarrow$ the integral is improper of the second type

We split the integral into two integrals around the problem point; $x=0$:

$x=0$ is the problem point

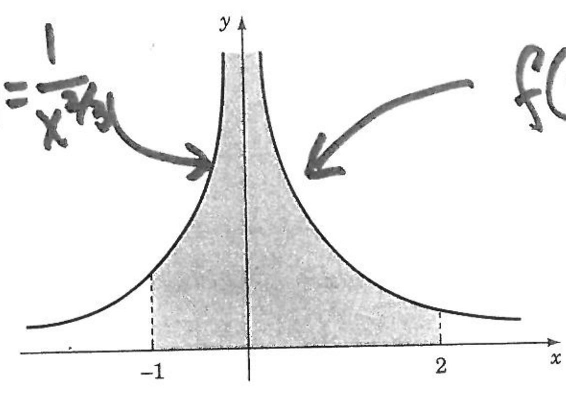
$$\int_{-1}^{2} \frac{1}{x^{2/3}}\, dx = \int_{-1}^{0} \frac{1}{x^{2/3}}\, dx + \int_{0}^{2} \frac{1}{x^{2/3}}\, dx$$

Now, we will use the same idea as above. Since it doesn't make sense to talk about $1/x^{2/3}$ at $x = 0$, we replace the 0 in the first integral by a limit.

$$\int_{-1}^{0} \frac{1}{x^{2/3}}\, dx = \lim_{b \to 0^-} \int_{-1}^{b} \frac{1}{x^{2/3}}\, dx$$

We take the limit as $b$ approaches 0 from the left, since the interval of integration is all to the left of 0. Then we can compute the integral and take

$f(x) = \frac{1}{x^{2/3}}$




$f(x) = \frac{1}{x^{2/3}}$

the limit:

$$\lim_{b\to 0^-}\int_{-1}^{b}\frac{1}{x^{2/3}}\,dx = \lim_{b\to 0^-} 3x^{1/3}\Big|_{-1}^{b} = \lim_{b\to 0^-}[3b^{1/3}]-[3(-1)^{1/3}] = 3$$

Similarly,

$$\lim_{b\to 0^+}\int_{b}^{2}\frac{1}{x^{2/3}}\,dx = \lim_{b\to 0^+} 3x^{1/3}\Big|_{b}^{2} = \lim_{b\to 0^+}[3(2^{1/3})]-[3b^{1/3}] = 3(2)^{1/3}$$

So

$$\int_{-1}^{2}\frac{1}{x^{2/3}}\,dx = 3+3(2)^{1/3}\approx 6.780 \Rightarrow \text{converges}$$

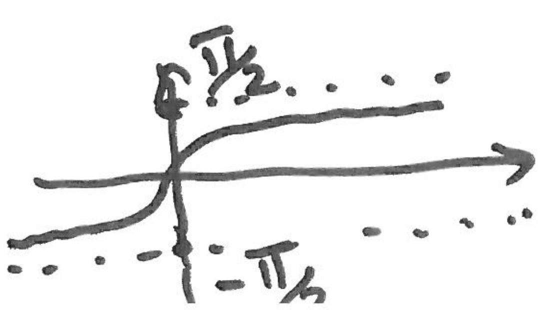
Example: • $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$

$$= \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_{0}^{\infty}\frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty}\Big[\arctan x\Big]_a^0 = 0-\left(-\frac{\pi}{2}\right)=\frac{\pi}{2}.$$

$$+$$

$$+ \lim_{b\to\infty}\Big[\arctan x\Big]_0^b = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$




$$= \underline{\underline{\frac{\pi}{2}+\frac{\pi}{2}}} = \pi$$

Example: see Sec. 5.10.

For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

Solution:

$$\int_1^\infty \frac{1}{x^p}\,dx := \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx .$$

where p is a parameter $\in \mathbb{R}$ (a constant)

Let us consider $p=1$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \Big[\ln|x|\Big]_1^b$$

$$= \lim_{b\to\infty} (\ln b - \ln 1)$$

$$= \infty \Rightarrow \text{divergent}$$

or the integral diverges

$$\Longrightarrow \int_1^\infty \frac{1}{x^p}\,dx \text{ diverges when } p=1.$$

$p \neq 1$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t \to \infty} \int_1^t x^{-p}\,dx$$

$$= \lim_{t \to \infty} \left. \frac{x^{-p+1}}{-p+1} \right]_{x=1}^{x=t}$$

$$= \lim_{t \to \infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right]$$

If $p > 1$ then $p - 1 > 0$

$\Rightarrow$ if $t \to \infty$, $t^{p-1} \to \infty$

$$\Rightarrow \frac{1}{t^{p-1}} \xrightarrow[t \to \infty]{} 0$$

converges

$$\Rightarrow \boxed{\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \text{ if } p > 1}$$

for $p \leq 1$ the integral diverges

11/11/2013

**Subject:** • Revisit: Integration by Parts

• Partial Fractions and their use in Integration

**Next:**

• Numerical Methods of Integration

• Improper Integrals $\int_a^\infty f\,dx$

Read: All Sections of Chapter 5.

# Decomposing a rational function into simpler rational functions

**Example 1:** $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$

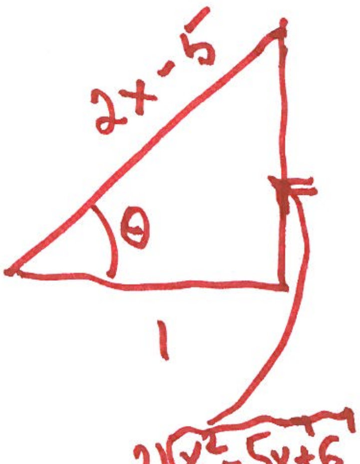
(i) Substitution Method

$$x^2-5x+6 = \left(x-\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

Let: $\quad x-\frac{5}{2} = \frac{1}{2}\sec\theta$

$$dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$$

$$\Rightarrow \int \frac{1}{x^2-5x+6}\,dx = \int \frac{\frac{1}{2}\sec\theta\tan\theta}{\left(\frac{1}{4}\right)\tan^2\theta}\,d\theta$$




$$= 2\int \csc\theta\,d\theta \quad \text{from the table}$$

$$= 2\ln\left|\csc\theta - \cot\theta\right| + C$$

$$= 2\ln\left|\frac{2x-5}{2\sqrt{x^2-5x+6}} - \frac{1}{2\sqrt{x^2-5x+6}}\right| + c$$

$$= 2\ln\left|\frac{x-3}{\sqrt{x^2-5x+6}}\right| + C$$

$$= \ln\left|\frac{(x-3)^2}{\underbrace{x^2-5x+6}_{(x-3)(x-2)}}\right| + C$$

$$= \underbrace{\ln|x-3| - \ln|x-2|} + C$$

Now

Suppose that you observed

$$\frac{1}{x^2-5x+6} = \frac{1}{x-3} - \frac{1}{x-2}$$

Partial Fraction Decomposition

$$\Rightarrow \int \frac{1}{x^2-5x+6}\,dx = \int\left(\frac{1}{x-3} - \frac{1}{x-2}\right)dx$$

$$= \ln|x-3| - \ln|x-2| + C$$

Checking: $\left[\ln|x-3| - \ln|x-2| + C\right]' = \frac{1}{x-3} - \frac{1}{x-2}$

$$= \frac{1}{x^2-5x+6}$$

## Study Tip:

$$\frac{1}{x-2} + \frac{-1}{x+3} = \frac{5}{(x-2)(x+3)}$$

reverse this process

- Recall from algebra:

"Every polynomial with real coefficients can be factored into linear and irreducible quadratic factors."

$$\begin{aligned} x^5 + x^4 - x - 1 &= x^4(x+1) - (x+1) \\ &= (x^4 - 1)(x+1) \\ &= (x^2+1)(x^2-1)(x+1) \\ &= (x^2+1)(x+1)(x-1)(x+1) \\ &= (x^2+1)(x-1)(x+1)^2 \end{aligned}$$

repeated linear factor

irreducible quadratic factor

a linear factor

Using this factorization you can write the partial fraction decomposition of the rational expression

$$\frac{N(x)}{x^5 + x^4 - x - 1}$$

a polynomial of degree less than 5

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{Dx+E}{x^2+1}$$

# Decomposition of $\frac{N(x)}{D(x)}$ into Partial Fractions:

degree $N \geq$ degree $D$

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)} \text{ ;}$$

- Linear factors;
- Quadratic factors.

$\boxed{\text{Review of factorization techniques}}$

Linear Factors:

Example 1: Distinct Linear Factors

$$\frac{1}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} \Big| (x-3)(x-2)$$

Multiplying by the least common denominator $(x-3)(x-2)$

$$1 = A(x-2) + B(x-3)$$ basic equation

To solve for A, let $x=3$

$$1 = A(3-2) + B(3-3)$$

$$1 = A(1) + B(0) \implies A = 1$$

To solve for B, let $x=2$

$$1 = A(2-2) + B(2-3)$$

$$1 = A(0) + B(-1)$$

$$B = -1 \implies$$

$A = 1$
$B = -1$

# Partial Fractions:

$$\int \frac{3x-1}{x^2+x-2}\,dx = ?$$

$$\frac{3x-1}{\underbrace{x^2+x-2}_{(x-1)(x+2)}} = \frac{A}{x-1} + \frac{B}{x+2} \quad \Big| \cdot (x^2+x-2)$$

$$3x-1 = A(x+2) + B(x-1)$$

$$3x-1 = (A+B)x + 2A - B$$

$$\Rightarrow \begin{cases} A+B = 3 \\ 2A-B = -1 \end{cases}$$

$$3A = 2 \Rightarrow \boxed{A = \frac{2}{3}}$$

$$\boxed{B = 3 - \frac{2}{3} = \frac{7}{3}}$$

$$\int \frac{3x-1}{x^2+x-2}\,dx = \int \frac{2/3}{x-1}\,dx + \int \frac{7/3}{x+2}\,dx$$

$$= \frac{2}{3}\ln|x-1| + \frac{7}{3}\ln|x+2| + C$$

<u>Note:</u>

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

- $$\int \underbrace{\frac{x^3-x^2-7x+2}{x^2-3x+2}}\,dx$$

$$= x + 2 + \frac{-3x-2}{x^2-3x+2}$$

$\Longrightarrow$ Read Sections: 5.6 + 5.7

11-8-2013

Subject: Integration by Parts
Section 5.6
Partial Fractions; Sec. 5.7

Next time:

- Trigonometric Integrals
- Integration Using Partial Fractions
- Integration Using Tables.

Recall:

$$[f(x)\cdot g(x)]' = f'(x)g(x) + f(x)g'(x)$$

$$\int \underline{\qquad\qquad} = \int \underline{\qquad\qquad} + \int \underline{\qquad}$$

$$\Rightarrow \quad f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$\Rightarrow \boxed{\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)}$$

Formula:

If $u = f(x)$, $\quad v = g(x)$

$du = f'(x)\,dx \qquad dv = g'(x)\,dx$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

or

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

## Examples:

1. Find $\int x \ln x \, dx$

Let's try $u = \ln x$ and $dv = x\,dx$

Make a chart:

$$u = \ln x \qquad dv = x\,dx$$

$$du = \frac{1}{x}\cancel{\ln x}\,dx \qquad v = \frac{x^2}{2}$$

we get v by integrating dv.

$$\Rightarrow \int x \ln x \, dx = \int u\,dv = uv - \int v\,du$$

$$= \underbrace{\frac{x^2}{2}\cdot \ln x}_{uv = vu} - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\cdot \ln x - \int \frac{x}{2}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

• Checking: $\left(\frac{x^2}{2}\ln x - \frac{x^2}{4} + C\right)' = x\ln x + \frac{x^2}{2}\cdot\frac{1}{x} - \frac{x}{2} = x\ln x$

• Example 2:

What about $\int 1\cdot \ln x \, dx$ ?

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$$\int \ln x \, dx = x \ln x - \int 1 \cdot dx$$

$$= x \ln x - x + C .$$

$$(x \ln x - x + C)' = 1 \cdot \ln x + x \cdot \cancel{\frac{1}{x}} - \cancel{1} + 0$$

How to decide what should be u and what should be dv ?

Example: $\int x e^x \, dx$

$u = \cancel{e^x}\ x \qquad dv = \cancel{x}\ e^x \, dx$

$du = \cancel{e^x} \, dx \qquad v = \cancel{\frac{x^2}{2}}\ e^x$

$$\int x e^x \, dx = \cancel{\frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x \, dx}$$ more difficult

$$= x e^x - \int e^x \, dx$$

$$\int x e^x \, dx$$

$u = x$ $\quad dv = e^x dx$

$du = dx$ $\quad v = e^x$

$$= xe^x - \int e^x \, dx = xe^x - e^x + C$$

Checking: $(xe^x - e^x + C)'$

$$= e^x + xe^x - e^x = xe^x$$

<u>Example 3:</u> $\int x \cos x \, dx$

$u = x$ $\quad dv = \cos x \, dx$

$du = dx$ $\quad v = \sin x$

$$= x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + C$$

Checking: $(x \sin x + \cos x + C)'$

$$= \sin x + x \cos x - \sin x = x \cos x$$

# Trigonometric Substitution:

Recall:

$$\boxed{\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}}$$

Example 1:

Calculate: $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$

let $x = \sin\theta$

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$dx = \cos\theta\, d\theta$

$$\Longrightarrow \int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\cos\theta}\cdot\cos\theta\, d\theta = \theta + C$$

$\theta = \arcsin x$

Example 2:

Calculate:

$$\int \frac{1}{4+x^2}\,dx = \int \frac{1}{4+4\tan^2\theta}\,d\theta$$

$$= \int \frac{1}{4(1+\tan^2\theta)}\,d\theta$$

Let $x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{1}{4\sec^2\theta}\,2\cdot\sec^2\theta\,d\theta$$

$$= \int \frac{1}{2}\,d\theta = \frac{1}{2}\theta + c$$

and

$$\theta = \arctan\left(\frac{x}{2}\right)$$

$$\Longrightarrow \int \frac{1}{4+x^2}\,dx = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c$$

Checking: $\left(\frac{1}{2}\arctan\left(\frac{x}{2}\right) + c\right)' = \frac{1}{2}\,\frac{1}{1+\frac{x^2}{4}}\cdot\frac{1}{2} = \frac{1}{4+x^2}$

$\longrightarrow \arcsin(x) \neq \frac{1}{\sin(x)}$

$$\int \underbrace{\sin^{-1}(t)}_{u}\,\underbrace{dt}_{dv}$$

$$u = \sin^{-1}(t) \qquad dv = dt$$

$$du = \frac{1}{\sqrt{1-t^2}}\,dt \qquad v = \int dt = t$$

$$= uv - \int v\,du$$

$$= \sin^{-1}(t)\cdot t - \boxed{\int t \cdot \frac{1}{\sqrt{1-t^2}}\,dt}$$

$$= \sin^{-1}(t)\,t + \frac{(1-t^2)^{1/2}}{\cancel{1/2}} + C$$